Non-Redundant Derivatives in a Dynamic General Equilibrium Economy∗

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Abstract

In this article, we consider an experiment where a non-redundant derivative is introduced in an economy where agents are heterogenous with respect to their risk aversion. This experiment allows us to study the effect of introducing a non-redundant derivative on the prices of more primitive securities, such as a stock and bond. It also allows us to examine the valuation of this non-redundant derivative and the role it plays in an investor’s dynamic portfolio strategy. We undertake this experiment by comparing to versions of a dynamic general equilibrium exchange economy where both the endowment and its growth rate are stochastic: in the first version, only a stock and a zero-supply instantaneously riskless bond are traded, while in the second version a derivative is also available. Our main contribution is to characterize in closed form (using asymptotic analysis) the equilibrium in these two versions of the economy. We find that the introduction of a derivative leads to an increase in the interest rate and the volatility of stock returns.

Keywords: Asset pricing, derivative valuation, portfolio choice, incomplete markets

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1 Introduction

In the Black and Scholes (1973) and Merton (1973) models of derivative pricing, the derivative is a redundant claim whose payoff can be replicated exactly using existing securities, the stock and the bond. In this paper, we take one step back from the economy considered by Black, Scholes and Merton, and study the effect of introducing a derivative in an economy with heterogeneous agents where the derivative is not a redundant security. In such a setting, Ross (1976), Hakansson (1978) and Arditti and John (1980) describe in the context of static models how the option will affect risk sharing and investment opportunities. Our objective in undertaking this exercise is to understand the effect of introducing a non-redundant derivative on the prices of already-existing securities, and in particular, on the volatility of stock returns, which can only be studied in a dynamic model. We also study the valuation of this non-redundant derivative and the role it plays in the dynamic portfolio strategies of investors.

In order to accomplish these goals, we consider a continuous-time, general-equilibrium endowment (exchange) economy with two agents who derive utility for consumption over their lifetime. These two agents differ in their degree of risk aversion (one has log utility and the other power utility), and therefore, trade financial claims for the purpose of risk sharing. In this economy, both the endowment and its growth rate are assumed to be stochastic. In the first version, only a stock and a zero-supply instantaneously riskless bond are available for trading, so that financial markets are incomplete. In the second version, we introduce a derivative that allows the agent to hedge perfectly the risk arising from the stochastic growth rate of the aggregate endowment. To understand the role of the derivative in an agent’s optimal investment strategy and its effect on the valuation of existing securities, we then compare portfolio policies and prices across these two versions of the economy. We find that the introduction of the derivative leads to an increase in the interest rate and the volatility of stock returns.

We now describe how our work is related to the existing literature. Closest to our work are the papers studying models with investors who differ in risk aversion. It is well-known that the Black, Scholes and Merton formula for option prices holds in an equilibrium framework only if there exists a representative investor with constant relative risk aversion (see, for example, Rubinstein (1976), Breeden and Litzenberger (1978), Brennan (1979), Stapleton and Subrahmanyam (1989), Bick (1987, 1990) and He and Leland (1993)). In order to
understand the pricing anomalies of the Black, Scholes and Merton formula, Bates (2001) considers an economy with investors who differ in their risk aversion and where crashes can occur in the market. In this economy, the less crash-averse investors insure the more risk-averse investors through options that complete the market. Weinbaum (2001) considers the effect of heterogeneity on redundant options in a dynamic exchange economy where investors derive utility only from terminal wealth, which is a special case of the complete-market economy considered in Wang (1996) and Kogan and Uppal (2001). In a single period (two-date) model, Niehaus (2001) studies numerically the effect of heterogeneity in the valuation of a non-redundant option, and compares this to a model where all agents are identical. A single-period model studying the effect of heterogeneity in risk aversion (and beliefs) on prices of redundant options is Benninga and Mayshar (2000).\textsuperscript{1} In contrast to this strand of the literature, we characterize analytically the dynamic equilibrium in both the complete and the incomplete market setting.

Our work is also related to the literature studying the demand for option-like payoffs in the context of portfolio insurance. Basak (1995) studies a general equilibrium model with two types of agents: “normal agents” and “portfolio insurers,” with the objective being to study the effects of portfolio insurance on the market price, volatility and risk premium. Grossman and Zhou (1996) consider a similar model but with consumption only on the terminal date. In these models, demand for options is generated directly from the utility function of portfolio insurers.\textsuperscript{2} Demand for derivatives in our model, however, arises endogenously from the desire to share risk. Moreover, in these models portfolio insurance is a redundant strategy, while the contingent claim in our analysis is not a redundant asset.

In addition to models where agents differ in their risk aversion, the literature also includes papers with other sources of heterogeneity. For instance, Franke, Stapleton, and Subrahmanyam (1998) consider a single-period model where agents have different non-hedgeable background risks leading to market incompleteness, and hence, a demand for insurance in the form of options on the marketable risks. Assuming exponential utility for agents, Calvet, Gonzalez-Eiras, and Sodini (2001) examine the effect on asset prices of introducing non-redundant options in a single-period model with heterogeneous random income

\textsuperscript{1}Other papers with investors who are heterogeneous with respect to risk aversion include the production-economy model in Dumas (1989) and the model where agents have catching-up-with-the-Joneses preferences in Chan and Kogan (2000); however, these papers do not consider the valuation of options.

\textsuperscript{2}See also Leland (1980) and Brennan and Solanki (1979).
and participation costs and Willen (1999) uses numerical methods to study a multiperiod model of incomplete markets but without participation costs.³

A large part of the theoretical literature studying the interaction between stock markets and options considers a framework where agents have divergent beliefs or asymmetric information. Leland (1980) studies the demand for options in a model where agents differ in expectations. In a mean-variance model where investors disagree about the volatility of future prices but agree on expected values, Detemple and Selden (1991) find that the option introduction leads to an increase in the stock price. Detemple (1990) shows that the introduction of options implies a lower equilibrium price for the stock. Other models studying the role of options in the presence of information asymmetries include Stein (1987), Grossman (1988, 1989), Huang and Wang (1997), and Back (1993). Gennette and Leland (1990) consider the effect of portfolio insurance on the level and volatility of prices under asymmetric information. In our model, we assume that all agents are perfectly informed.

There is a large literature that documents empirically the effect of prices following the introduction of derivatives. For instance, Branch and Finnerty (1981), Conrad (1989) and Detemple and Jorion (1990) find positive excess returns around call option introduction but no price effects at the time of announcement of the option introduction. In more recent work, Sorescu (2000) finds that this positive price effect is limited to options introduced prior to 1981, and after that cut-off date the price effect is negative. Jochum and Kodres (1998) provide international evidence. A comprehensive survey of the empirical literature is presented in Mayhew (2000).

The role of options in portfolio strategies has been discussed in Merton, Scholes, and Gladstein (1978, 1982) who consider how call and put-options can affect the risk-return tradeoff for portfolio strategies. More recently, Liu and Pan (2001) consider the role of options in a partial equilibrium setting where the underlying stock is specified exogenously to have stochastic volatility and price jumps. Our model will allow us to address these questions in a general equilibrium setting that has the natural advantage of internal consistency, in the sense that prices are determined endogenously.⁴

³Leisen and Judd (2000) consider the role of options in hedging income risks in a partial equilibrium model where the stock and bond prices are given exogenously.

⁴Bailey and Stulz (1989) illustrate the importance of choosing an interest rate and volatility process that is internally consistent in the context of valuing options in a general equilibrium model where there is only a representative investor and the option is redundant.
We now describe how the approach we use to characterize the equilibrium analytically is related to existing methods for obtaining closed-form solutions. One stream of the literature—for example, Liu (1998) and Wachter (1998) in partial equilibrium and Wang (1996) in general equilibrium—assumes that financial markets are complete, and then uses the martingale technique of Cox and Huang (1989) to determine the optimal consumption and portfolio rules. A second branch—for instance, Kim and Omberg (1996) and Liu (1998)—assumes that investors derive utility only from terminal wealth and not from intermediate consumption. Then, under a particular specification of the investment opportunity set, this assumption allows them to solve the dynamic programming equation in closed form. A third approach, developed by Campbell (1993) allows for intermediate consumption and incomplete financial markets, but chooses a convenient specification for the evolution of the investment opportunity set, and then makes appropriate approximations in order to overcome the non-linearity of the problem. Finally, Kogan and Uppal (2001) use perturbation analysis to solve for the optimal portfolio policies and equilibrium prices. Their technique relies on the insight that an investor with log utility holds only a myopic portfolio, and therefore, in this case it is straightforward to identify the solution. Perturbing the solution of the log investor is then sufficient to get the solution (up to first order in the degree of risk aversion) for investors with risk aversion close to unity.

None of these approaches is suitable for the analysis we wish to undertake. For instance, the first approach can be used only if financial markets are complete, while the other two approaches work only for particular specifications for the evolution of the investment opportunity set. Given that we wish to solve for the optimal policies when financial markets are not complete, the first method cannot be applied to solve our problem. Also, since we are considering a general equilibrium model, the evolution of the state variables cannot be specified exogenously, which renders the second and third approaches unsuitable for our purpose. The Kogan and Uppal (2001) approach could be used in our setting to obtain the solution for first-order (in risk aversion), but because the differences between the economies with complete and incomplete markets arises only at the level of higher order terms, this approach needs to be extended.

The solution method we develop combines the strategy used in Dumas (1989) to study (numerically) the equilibrium in a production economy where there are two agents one of whom has log utility, with the perturbation approach described in Kogan and Uppal (2001). Characterizing the equilibrium in an economy with two agents requires solving the Hamilton-Jacobi-Bellman partial differential equation for each of the two agents simultaneously so that the market clearing conditions are satisfied. If one of the agents has log utility, then the form of the optimal consumption and portfolio policies of this agent are known, which Dumas uses along with the market-clearing conditions, to reduce the problem to one where only the remaining non-linear differential equation needs to be solved. We repeat the same steps in the context of the exchange economy that we are studying, and then to obtain an analytic (but approximate) solution to the non-linear differential equation we expand the entire Bellman equation in terms of the risk aversion parameter in the spirit of Kogan and Uppal, which allows us to obtain first- and higher-order expressions for the equilibrium policies and prices. Details of this method are provided in the appendix.

We should highlight that the method used to obtain closed-form results relies on asymptotic analysis; hence, the conclusions are valid only locally, in the region where the relative risk aversion of the agent is close to unity. While we provide higher-order expressions for the equilibrium portfolio policies and prices, we do not have general results about the global convergence of these quantities.

The rest of the paper is arranged as follows. In Section 2, we describe the basic model of a heterogeneous agent exchange economy where the primitive financial assets, a stock and a zero-net supply instantaneously riskless bond, are not sufficient to complete the financial market. We then describe the method used to characterize analytically the equilibrium in this economy. In Section 3, we introduce an additional financial claim, a zero-net supply option, that completes the market and then characterize the equilibrium in this economy. In Section 4, we compare the portfolio policies and prices in the economy with incomplete markets to the economy where financial markets are complete. We conclude in Section 5 with a summary of our results and a discussion of the limitations of our analysis. Intermediate results are presented in lemmas while the main results are highlighted in propositions; the details of the derivations for all the results are presented in the appendix.
2 The economy without the derivative

In this section, we consider an exchange economy where the endowment and its growth rate are stochastic. There are two agents in the economy who differ only in their degree of risk aversion. One agent is assumed to have relative risk aversion of unity (log utility) while the other agent has power utility. In this section, we assume that the only traded securities are the stock, which is a claim on aggregate endowment, and an instantaneously riskless bond, which is in zero net supply. Given that there are two sources of risk, the financial market with just a stock and a bond is not complete. Below, we first describe these features of the exchange economy in detail and then derive the optimal consumption and portfolio rules for the two agents and the equilibrium expected return and volatility of the return on the risky asset along with the interest rate.

2.1 The exchange economy

In this section, we describe the features of the model: the preferences of agents, the financial assets that they can choose to hold, and the stochastic nature of this investment opportunity.

2.1.1 Preferences

The utility functions of the two agents are assumed to be time separable. The utility of the first agent is given by

$$\beta E_t \left[ \int_t^{\infty} e^{-\beta(s-t)} \log C^*_s ds \right],$$

where $C^*_t$ is the flow of consumption, $\beta$ is the rate of time preference and relative risk aversion and the elasticity of intertemporal substitution are both unity. The utility of the second agent is given by a power function

$$\beta E_t \left[ \int_t^{\infty} e^{-\beta(s-t)} \frac{C^\alpha_s}{\alpha} ds \right].$$

Note that $\beta$ is the subjective rate of time preference, relative risk aversion is given by $1 - \alpha$, the elasticity of intertemporal substitution is given by $1/(1 - \alpha)$, $C_t$ is the flow of consumption, and below we will use the notation $c_t \equiv C_t/W_t$.

We take the unusual step of pre-multiplying expected lifetime utility by $\beta$ in order make our results directly comparable with those for the case where the second agent is assumed to have recursive utility.
2.1.2 Financial assets

We assume that the agents can choose to invest in either a short-term riskless asset with rate of return \( r_t \) or a risky asset (dividend paying stock) with price \( P_t \) with cumulative returns described by the process

\[
\frac{dP_t + e_t dt}{P_t} = \mu_{R_t} dt + \sigma_{P_t} dZ_{P_t}.
\]

(3)

Here \( \mu_{R_t} \) is the instantaneous expected return and \( \sigma_{P_t} \) is the volatility. Note our convention where we denote all stochastic variables with the subscript \( t \). The stock is a claim on the aggregate endowment, whose evolution is given by the following stochastic process:

\[
d e_t = (\mu + X_t) e_t dt + \sigma_e e_t dZ_{et},
\]

(4)

with

\[
d X_t = \mu_X dt + \sigma_X dZ_{Xt},
\]

(5)

\[
[dZ_{et}, dZ_{Xt}] = \rho e X dt,
\]

(6)

and where \( \sigma_e, \mu_X, \sigma_X \) and \( \rho e X \) are constant parameters. The process for the state variable \( X_t \) dictates the instantaneous expected growth rate of the aggregate endowment process. \(^7\) For expository clarity, we consider the special case where

\[
\mu_X = 0 \quad \text{and} \quad \rho e X = 0.
\]

Given the above assumptions, the conditional expected growth rate of endowment is given by:

\[
\Gamma \equiv \mu + X - \frac{1}{2} \sigma_e^2.
\]

2.1.3 Investment opportunity set

The state of the economy is given by

\[
X_t = \begin{pmatrix} X_t \\ \omega_t \end{pmatrix},
\]

where, \( \omega_t = W_t / (W^*_t + W_t) \). The evolution of \( X_t \) is given exogenously by (5) and that of \( \omega_t \) by:

\[
d \omega_t = \mu_{\omega t} dt + \sigma_{\omega t} dZ_{\omega t},
\]

(7)

\(^7\)With a constant expected growth rate, this model reduces to the complete-market economy studied in Wang (1996) and Kogan and Uppal (2001).
where the drift and the diffusion coefficients are functions of the state: $\mu_{\omega t} = \mu_\omega (X_t)$ and $\sigma_{\omega t} = \sigma_\omega (X_t)$ and are determined endogenously in equilibrium. Expanding the coefficients for the process in (7) in powers of $\alpha$ gives:

\[
\begin{align*}
\mu_\omega (X) &= \sum_{i=0}^{N} \mu_{\omega,i} (X) + O \left( \alpha^{N+1} \right), \\
\sigma_\omega (X) &= \sum_{i=0}^{N} \sigma_{\omega,i} (X) + O \left( \alpha^{N+1} \right).
\end{align*}
\]

When $\alpha = 0$, both types of agents have logarithmic preferences. As a result, the cross-sectional distribution of wealth in the economy does not change over time and $\omega_t$ is constant, which implies that $\mu_{\omega,0} (X) = 0$ and $\sigma_{\omega,0} (X) = 0$.

### 2.2 Individual consumption and portfolio policies

From the first-order conditions of the Bellman equation, the optimal consumption and portfolio policies for the two types of investors are as follows.

**Lemma 1** *The optimal policies are:*

\[
\begin{align*}
\hat{c}^* (X, \omega) &= \beta \\
\hat{c} (X, \omega) &= \left[ \beta^{-1} \exp [\alpha g (X, \omega)] \right]^{\frac{1}{\alpha - 1}} \\
\theta^* (X, \omega) &= \frac{\mu_R (X, \omega) - r (X, \omega)}{\sigma_P (X, \omega)} \\
\theta (X, \omega) &= \frac{1}{1 - \alpha} \frac{\mu_R (X, \omega) - r (X, \omega)}{\sigma_P (X, \omega)} \\
&\quad + \frac{\alpha}{1 - \alpha} \left[ \frac{1}{\sigma_P^2 (X)} \left( \frac{\sigma_{P_X} (X, \omega)}{\sigma_P (X, \omega)} \right) \cdot \left( \frac{\partial g (X, \omega)}{\partial (X, \omega)} \right) \right],
\end{align*}
\]

where $\sigma_{P_X} (X, \omega)$ and $\sigma_{P_\omega} (X, \omega)$ are the covariances of stock returns with changes in the state variables $X$ and $\omega$.

Asymptotically, the function $g(X, \omega)$ which defines the value function of the agent with power utility can be expanded as

\[
g (X, \omega; \alpha) = g_0 (X) + \alpha g_1 (X, \omega) + O \left( \alpha^2 \right),
\]
where \( g_0 \) corresponds to the value function of the logarithmic investor in a homogeneous-agent economy in which all investors have logarithmic preferences. The precise form of \( g_0 (X) \) and the higher-order expansions of the value function \( g(X, \omega) \), which can be obtained by expanding the Bellman equation, are stated in the following lemma.

**Lemma 2** The functions \( g_0 \) and \( g_1 \) are:
\[
g_0 (X) = \frac{\mu + X - \frac{1}{2} \sigma^2_e}{\beta} + \log \beta \tag{12}
\]
\[
g_1 (X, \omega) = \frac{\beta (\mu + X - \frac{1}{2} \sigma^2_e)^2 - 2 \omega \beta^2 (\mu + X - \frac{1}{2} \sigma^2_e) + 2 \sigma^2 \chi + \beta^2 \sigma^2_e}{2 \beta^3}. \tag{13}
\]

The value function of the logarithmic agent is given by
\[
\log W^* + g^{\log} (X, \omega).
\]

We can express \( g^{\log} (X, \omega) = g_0^{\log} (X, \omega) + \alpha g_1^{\log} (X, \omega) + O (\alpha^2) \), with the terms in the expansion presented in the following lemma.

**Lemma 3** The functions \( g_0^{\log} \) and \( g_1^{\log} \) are:
\[
g_0^{\log} (X) = \frac{\mu + X - \frac{1}{2} \sigma^2_e}{\beta} + \log \beta, \tag{14}
\]
\[
g_1^{\log} (X, \omega) = -\frac{\omega}{\beta} \left( \mu + X - \frac{1}{2} \sigma^2_e \right). \tag{15}
\]

The details of how one can determine the higher-order terms in the expansion of \( g \) and \( g^{\log} \) are given in the appendix.

### 2.3 Equilibrium in the economy without the derivative

The equilibrium in this economy is defined by the stock price process, \( P_t \), the interest rate process \( r_t \), the portfolio policies \( \{ \theta_t^*, \theta_t \} \) and the consumption processes \( \{ C_t^*, C_t \} \), such that: (i) given the price processes for financial assets, the consumption and portfolio choices are optimal for the agents, (ii) the goods market and the markets for the stock and the bond clear.
2.3.1 The market-clearing conditions

The market-clearing condition in the commodity market is

\[ C + C^* = e \]

or \( cW + c^*W^* = e \). Given that the aggregate financial wealth in the economy equals the total wealth of the stock market, \( W + W^* = P \), this can also be written as \( p^{-1} = c^* (1 - \omega) + c\omega \), where \( p \) is the stock price per unit aggregate endowment, \( p = \frac{P}{e} \). The market-clearing condition in the stock market is

\[ \omega \theta + (1 - \omega) \theta^* = 1. \]

Using the market-clearing condition, the expressions for \( g_0 \) and \( g_1 \) and defining

\[
\begin{align*}
ARA &= 1 - \omega \alpha \\
VRA &= \omega (1 - \omega) \alpha^2 \\
ART &= \frac{1 - \alpha (1 - \omega)}{1 - \alpha},
\end{align*}
\]

where ARA is the wealth-weighted average relative risk aversion in the economy, VRA is the variance of relative risk aversion in the economy and ART is the average risk tolerance in the economy, one obtains the following characterization of the equilibrium in this economy.

**Proposition 1** For the economy without the derivative, in equilibrium:

(i) The consumption policies of the two types of investors are

\[
c^* (X, \omega) = \beta,
\]

\[
c (X, \omega) = \beta + (ARA - 1) \left( \mu + X - \frac{1}{2} \sigma^2 e \right) \\
- \alpha^2 \left[ \mu + X - \omega \left( \mu + X - \frac{1}{2} \sigma^2 e \right) + \frac{\sigma^2 X}{\beta^2} \right] + O \left( \alpha^3 \right).
\] (17)

(ii) The optimal portfolio policies of the two types of investors are

\[
\theta^* (X, \omega) = \frac{1}{ART} - \frac{\omega^2 \sigma^2 X}{\beta^2 \sigma^2} \alpha^2 + O \left( \alpha^3 \right)
\] (18)

\[
\theta (X, \omega) = \frac{1}{(1 - \alpha) ART} + \frac{\omega (1 - \omega) \sigma^2 X}{\beta^2 \sigma^2} \alpha^2 + O \left( \alpha^3 \right)
\] (19)
where the hedging demand $H$ of the power utility investor is given by

$$H = \alpha^2 \frac{\omega^2}{\beta^2 \sigma_e^2} + O \left( \alpha^3 \right). \quad (20)$$

(iii) The stock price process is given by

$$P_t = p(X_t, \omega_t) e_t, \quad (21)$$

where

$$p(X, \omega) = \frac{1}{\beta} + \frac{1}{\beta^2} (1 - ARA) \Gamma + \alpha^2 \omega \beta \left[ 2 \Gamma (\Gamma - \beta) \omega + \beta \sigma_e^2 \right] + 2 \omega \sigma_X^2 + O \left( \alpha^3 \right) \quad (22)$$

$$\frac{dP_t + e_t dt}{P_t} = \mu_R(X, \omega) dt + \sigma_P(X, \omega) dZ_P(X, \omega), \quad (23)$$

where

$$\mu_R(X, \omega) = X + \beta + (ARA - 1) \Gamma + \alpha^2 \omega \beta \left[ -\beta^2 \sigma_e^2 + 2 \Gamma (\omega - 1) \beta (\beta - \sigma_e (1 + \sigma_e)) + 2 (\omega - 1) \sigma_X^2 \right] + O \left( \alpha^3 \right) \quad (24)$$

$$\sigma_P(X, \omega) = \sigma_e + VRA \frac{\sigma_e}{\beta} \Gamma + \frac{\alpha^2 \omega \sigma_X^2}{2 \beta^2 \sigma_e} + O \left( \alpha^3 \right) \quad (25)$$

$$dZ_P(X, \omega) = dZ_e + (1 - ARA) \frac{\beta \sigma_X}{\sigma_e} dZ_X \quad (26)$$

(iv) The interest rate is

$$r_t = \Gamma + \beta - \frac{\sigma_e^2}{2} + (ARA - 1) \left( \Gamma - \sigma_e^2 \right) + \alpha^2 \omega \left[ (1 - 2 \omega) \beta^2 \sigma_e^2 + 2 \Gamma (\omega - 1) \beta [\beta + \sigma_e (\sigma_e - 1)] + 2 (\omega - 1) \sigma_X^2 \right] + O \left( \alpha^3 \right) \quad (27)$$

(v) The cross-sectional wealth distribution evolves according to

$$d\omega_t = \mu_\omega dt + \sigma_\omega dZ_\omega,$$
where

\[
\mu_\omega = -\omega ((1 - \alpha) \omega - ARA) \sigma_e \\
+ VRA \left\{ \Gamma (1 - \omega) + (1 - 2\omega) \frac{\sigma_e^2}{2} + \frac{\sigma_X^2}{\beta^2} \right\} + O(\alpha^3) \tag{28}
\]

\[
\sigma_\omega = -\omega ((1 - \alpha) \omega - ARA) \sigma_e + VRA \left[ (1 - \omega) \sigma_e + \omega \frac{\sigma_X^2}{\beta^2 \sigma_e} \right] + O(\alpha^3) \tag{29}
\]

\[
dZ_\omega = dZ_P. \tag{30}
\]

(vi) The risk premium is given by

\[
RP (X, \omega) = \left[ ARA + VRA \frac{(2\Gamma - \beta)}{\beta} \right] \sigma_e^2 + O(\alpha^3)
\]

(vii) The Sharpe Ratio is given by

\[
SR (X, \omega) = \left[ ARA + VRA \frac{(\Gamma - \beta)}{\beta} \right] \sigma_e - \frac{\alpha^2 \omega^2}{2 \beta^2 \sigma_e} \sigma_X^2.
\]

In the rest of this section, we discuss the results stated in the above proposition.

2.3.2 Analysis of portfolio policies

The average risk tolerance in the economy fluctuates over time in response to the aggregate endowment shocks. From (18) and (19), we see that the fraction of total wealth held by each investor in the stock is the ratio of her individual risk tolerance to the average risk tolerance plus higher order terms. Hence, the more risk averse investor invests less heavily in the stock market.

The hedging demand shows how the non-log investor hedges against uncertainty in expected dividend growth. For the investor with power utility, the hedging demand from (19) and (20), is given by

\[
\frac{\alpha^2 \sigma_X^2}{\beta^2 \sigma_e^2} \omega + O \left( \alpha^3 \right).
\]

Note that

\[
\rho_{PX} = \alpha \beta \omega \frac{\sigma_X}{\sigma_e} + O \left( \alpha^2 \right).
\]

Hence, we can see that as the stock price becomes more highly correlated with \( X \) (the drift in the dividend growth rate), the more the stock will be used to hedge against uncertainty in future dividend growth.
2.3.3 Financial variables

From (22), the stock price per unit endowment or the price-dividend ratio, \( p \), is decreasing in average risk aversion. We can explain this from our portfolio choice results. The more risk averse investor is less heavily invested in the stock market. Hence, when the stock market rises, the less risk averse investor increases his share of aggregate wealth, decreasing the aggregate risk aversion. As the aggregate risk aversion drops, expected stock returns drop. Note that as the variance of average risk aversion increase stock holdings become more dispersed, increasing the volatility of stock returns.

The risk premium is increasing in the average relative risk aversion of the economy and also in it’s variance, if \( \Gamma > \beta/2 \). Similarly the Sharpe ratio is increasing in the average relative risk aversion of the economy and also in it’s variance, if \( \Gamma > \beta \). Note however that the Sharpe ratio decreases as the volatility of the drift of the growth rate of the economy increases.

The volatility of stock returns is a second moment and hence is independent of the average relative risk aversion, but is increasing in the variance of risk aversion.

As average risk aversion rises, so does the equilibrium interest rate, indicating that the bond price is decreasing in average risk aversion. The open interest in the bond market is given by

\[
OI_t = \frac{1}{2} \left( |1 - \theta_t| \omega_t + |1 - \theta_\ast| (1 - \omega_t) \right)
\]

\[
= \alpha (1 - \omega_t) \omega_t + O \left( \alpha^2 \right).
\]

Hence, the open interest in the bond market is a maximum when aggregate wealth is evenly distributed between agent types (that is, \( \omega = 1/2 \)).

3 The economy with the non-redundant derivative

In the general equilibrium exchange economy analyzed in Section 2, financial markets are incomplete. We now introduce a zero-net-supply derivative in that economy and then analyze the optimal consumption and portfolio policies in the presence of this non-redundant derivatives. The comparison of the equilibrium in this economy with that of the previous section is presented in Section 4.
3.1 Description of the non-redundant derivative

Let \( Q_t \) denote the price of the contingent claim which evolves according to:
\[
\frac{dQ_t}{Q_t} = \mu_t dt + \sigma_t dZ_t.
\]

We shall assume for analytical simplicity that \( \sigma_t dZ_t = dZ_X \). This derivative completes the market, so this assumption will not affect the equilibrium prices, but does make them easier to derive. Note that this derivative is not a redundant security, so we cannot use the Black-Scholes-Merton approach to value it; instead, its value will be determined by the market-clearing condition that must hold in equilibrium.

3.2 Individual consumption and portfolio policies

In the presence of the non-redundant derivative, the investor’s dynamic budget constraint is
\[
dW = [r + \theta_P (\mu_R - r) + \theta_Q (\mu_Q - r) - c] W dt + (\theta_P \sigma_P dZ_P + \theta_Q \sigma_Q dZ_Q) W.
\]

The optimal consumption and portfolio policies for the investor with power utility are given by the solution to the following Bellman equation:
\[
0 = \sup_{C,\theta_P,\theta_Q} \left\{ f(C, J) + \left( r + \theta_P (\mu_R - r) + \theta_Q (\mu_Q - r) - c \right) W J_W + \mu_X J_X \right. \\
+ \left. \frac{1}{2} \left( \theta_P^2 \sigma_P^2 + 2 \rho_{PQ} \theta_P \theta_Q \sigma_P \sigma_Q + \theta_Q^2 \sigma_Q^2 \right) W^2 J_W W \\
+ \left( \theta_P \sigma_P \rho'_{PX} + \theta_Q \sigma_Q \rho'_{QX} \right) \sigma_X W J_W X + \frac{1}{2} \sigma_X J_X X \sigma_X \right\}. \tag{31}
\]

We can now write down the optimal consumption and portfolio rules for the investors.

**Lemma 4** The exact optimal consumption policy for the logarithmic investor and power-utility investor are given by
\[
e^* = \beta \\
e = \left( \beta^{-1} \exp \left[ \alpha g(X, \omega) \right] \right)^{1/t},
\]
while the optimal portfolio policies for the log investor are
\[
\theta^*_P = \sigma_P^{-1} \left( 1 - \rho_{PQ}^2 \right)^{-1} \left[ \frac{\mu_R - r}{\sigma_P} - \rho_{PQ} \frac{\mu_Q - r}{\sigma_Q} \right]
\]
\[ \theta_Q^* = \sigma_Q^{-1} \left( 1 - \rho_{PQ}^2 \right)^{-1} \left[ \frac{(\mu_Q - r)}{\sigma_Q} - \rho_{PQ} \left( \frac{\mu_R - r}{\sigma_P} \right) \right] \]

and the portfolio rules for the investor with power utility are:

\[ \theta_P = (1 - \alpha)^{-1} \left( 1 - \rho_{PQ}^2 \right)^{-1} \sigma_P^{-1} \left[ \frac{(\mu_R - r)}{\sigma_P} - \rho_{PQ} \left( \frac{\mu_Q - r}{\sigma_Q} \right) + \alpha (\rho_{RX} - \rho_{PQ} \rho_{RX}) \sigma_X g'(X) \right] \]

\[ \theta_Q = (1 - \alpha)^{-1} \left( 1 - \rho_{PQ}^2 \right)^{-1} \sigma_Q^{-1} \left[ \frac{(\mu_Q - r)}{\sigma_Q} - \rho_{PQ} \left( \frac{\mu_R - r}{\sigma_P} \right) + \alpha (\rho_{RX} - \rho_{PQ} \rho_{RX}) \sigma_X g'(X) \right] \]

As before, one can obtain the zeroth-order expansion of the value function, while higher order terms in the expansion for \( g(X, \omega) \) are obtained by substituting the expressions for endogenous variables such as the interest rate into the Bellman equation.

**Lemma 5** The functions \( g_0 \) and \( g_1 \) corresponding to the expansion of the value function of the investor with power utility in the economy with the derivative is given by

\[ g_0(X) = \frac{\mu + X - \frac{1}{2} \sigma_e^2}{\beta} + \log \beta, \]

\[ g_1(X, \omega) = \frac{\beta^2 \sigma_e^2 + 2 \sigma_X^2 - 2 \omega \beta^2 \left( \mu + X - \frac{1}{2} \sigma_e^2 \right) + \beta \left( \mu + X - \frac{1}{2} \sigma_e^2 \right)^2}{2 \beta^3}. \]

Just as before, one can obtain higher-order terms of this expansion using the procedure described in the appendix.

### 3.3 Equilibrium in the economy with the derivative

The equilibrium is defined by the stock price and contingent claim price process, \( P_t, Q_t \), the interest rate process \( r_t \), the portfolio policies \( \{\theta_P^*, \theta_P, \theta_Q^*, \theta_Q\} \) and the consumption processes \( \{C^*_t, C_t\} \), such that: (i) given the price processes for financial assets, the consumption and portfolio choices are optimal for the agents; and (ii) the markets for the commodity, the stock, the bond and the contingent claim clear.

#### 3.3.1 The market-clearing conditions

Just as in the economy without the option, the market-clearing condition in the commodity market is

\[ C + C^* = e. \]
The market clearing conditions in the stock and option markets respectively are

\[
\begin{align*}
\theta_P \omega + \theta_P^* (1 - \omega) &= 1 \\
\theta_Q \omega + \theta_Q^* (1 - \omega) &= 0.
\end{align*}
\]

Using these market-clearing conditions we get the following result.

**Proposition 2** For the economy with the derivative, in equilibrium:

(i) The optimal consumption policies of the two types of investors are

\[
\begin{align*}
c^* (X, \omega) &= \beta, \\
c (X, \omega) &= \sum_{i=0}^{2} \alpha^i c_i + O \left( \alpha^3 \right),
\end{align*}
\]

where

\[
\begin{align*}
c_0 &= \beta \\
c_1 &= -\Gamma \\
c_2 &= - \left( \mu + X + \frac{\sigma_X^2}{\beta^2} \right) + \omega \Gamma.
\end{align*}
\]

(ii) The optimal portfolio policies of the two types of investors are

\[
\begin{align*}
\theta_P^* (X, \omega) &= \frac{1}{ART} + \frac{\omega^2 (1 - \omega)}{\beta} \Gamma \alpha^3 + O \left( \alpha^4 \right) \\
\theta_P (X, \omega) &= \frac{1}{(1 - \alpha) ART} - \frac{\omega (1 - \omega)^2}{\beta} \Gamma \alpha^3 + O \left( \alpha^4 \right) \\
\theta_Q^* (X, \omega) &= - \frac{\omega \sigma_X}{\beta} \alpha - \frac{\alpha^2 \omega}{\beta^2} \left[ \Gamma + (1 - 2 \omega) \beta \right] \sigma_X + O \left( \alpha^3 \right) \\
\theta_Q (X, \omega) &= \frac{(1 - \omega) \sigma_X}{\beta} \alpha + \frac{\alpha^2}{\beta^2} \sigma_X \left[ \Gamma + (1 - \omega) \beta \right] + O \left( \alpha^3 \right).
\end{align*}
\]

The hedging demands of the power utility investor for the stock and derivative are given by \( H_P \) and \( H_Q \) respectively, where

\[
\begin{align*}
H_P &= - \frac{\omega (1 - \omega)}{\beta} \Gamma \alpha^3 + O \left( \alpha^4 \right) \\
H_Q &= \frac{\sigma_X}{\beta} \alpha + \frac{\alpha^2}{\beta^2} \left[ \Gamma + (1 - \omega) \beta \right] \sigma_X + O \left( \alpha^3 \right).
\end{align*}
\]
(iii) The stock price is

\[ P_t = p(X_t, \omega_t) e_t, \]  
\[ p(X, \omega) = \frac{1}{\beta} + \frac{1}{\beta^2} (1 - ARA) \Gamma \]  
\[ + \alpha^2 \omega \beta \left[ \frac{2 \Gamma ((1 - \beta) \omega + \beta + \beta \sigma_e^2) + 2 \omega \sigma_X^2}{2 \beta^4} \right] + O (\alpha^3) \]  
\[ \frac{dP_t + e_t dt}{P_t} = \mu_R (X, \omega) dt + \sigma_P (X, \omega) dZ_P (X, \omega), \]  

where

\[ \mu_R (X, \omega) = X + \beta + (ARA - 1) \Gamma \]  
\[ + \frac{\omega}{2 \beta} \left[ 2 \Gamma (1 - \omega) \left( \sigma_e^2 + \sqrt{\sigma_e^2 + \frac{\sigma_X^2}{\beta^2}} - \beta \right) \right] + O (\alpha^3) \]  
\[ \sigma_P (X, \omega) = \sigma_e + VR A \left( \frac{\sigma_X}{\beta} \Gamma + \frac{\alpha^2 \omega \sigma_X^2}{2 \beta^2 \sigma_e} \right) + O (\alpha^3). \]  

(iv) The derivative price is given by \( Q_t \)

\[ \frac{dQ_t}{Q_t} = \mu_Q (X, \omega) dt + \sigma_Q (X, \omega) dZ_Q (X, \omega) \]

where

\[ \mu_Q = \beta + \frac{\sigma_e^2}{2} + ARA \left( \Gamma - \sigma_e^2 \right) - \alpha^2 \omega \sigma_e^2 \]  
\[ + VR A \left( \frac{\Gamma}{\beta} \left( \sqrt{\sigma_e^2 + \frac{\sigma_X^2}{\beta^2}} - \sigma_e^2 - \beta \right) + \frac{\sigma_e^2}{2} \right) + O (\alpha^3). \]  

(v) The interest rate is

\[ r_t = \Gamma + \beta - \frac{\sigma_e^2}{2} + (ARA - 1) \left( \Gamma - \sigma_e^2 \right) - \frac{1}{2} \alpha^2 \omega^2 \sigma_e^2 \]  
\[ + VR A \left( \frac{1}{2} \sigma_e^2 - \frac{\Gamma [\beta + \sigma_e (\sigma_e - 1)]}{\beta} - \frac{\sigma_X^2}{\beta^2} \right) + O (\alpha^3). \]  

(vi) The cross-sectional wealth distribution evolves according to

\[ d\omega_t = \mu_\omega dt + \sigma_\omega dZ_\omega, \]
where

\[ \mu = -\omega ((1 - \alpha) \omega - ARA) \sqrt{\sigma_e^2 + \sigma_X^2} \]
\[ + VRA \left\{ \Gamma (1 - \omega) + (1 - 2\omega) \frac{\sigma_e^2}{2} + (1 - \omega) \frac{\sigma_X^2}{\beta^2} \right\} + O(\alpha^3) \] (37)

\[ \sigma = -\omega ((1 - \alpha) \omega - ARA) \sqrt{\sigma_e^2 + \sigma_X^2} \]
\[ + VRA \left[ \frac{\Gamma \sigma_X^2 + (1 - \omega) \beta \left( \frac{\sigma_e^2}{2} + \frac{\sigma_X^2}{\beta^2} \right)}{\beta \sqrt{\sigma_e^2 + \frac{\sigma_X^2}{\beta^2}}} \right] + O(\alpha^3) \] (38)

\[ dZ = \frac{\beta \sigma_e dZ_e + \sigma_X dZ_X}{\beta \sqrt{\sigma_e^2 + \frac{\sigma_X^2}{\beta^2}}} + \alpha \Gamma \sigma_e \sigma_X \frac{(\beta \sigma_e dZ_X - \sigma_X dZ_X)}{\beta (\beta \sigma_e^2 + \sigma_X^2) \sqrt{\sigma_e^2 + \frac{\sigma_X^2}{\beta^2}}} + O(\alpha^2) \] (39)

(vii) The risk premium on the stock is given by

\[ RP_{\text{Stock}}(X, \omega) = \left[ ARA + VRA \frac{(2 \Gamma - \beta)}{\beta} \right] \sigma_e^2 + O(\alpha^3) \]

(viii) The risk premium on the option is given by

\[ RP_{\text{Option}} = -\alpha^3 \frac{\omega \sigma_X}{2 \beta^4 \sigma_e^2} \left[ 2 \Gamma^2 \beta \sigma_e^2 + 4 \Gamma (1 - \omega^2) \beta^2 \sigma_e^2 + 2 (1 - \omega)^2 \beta^3 \sigma_e^2 - \omega^2 \beta \sigma_X^2 + 6 \sigma_e^2 \sigma_X^2 \right] \]
\[ + O\left(\alpha^3\right) \]

(ix) The Sharpe Ratio on the stock is given by

\[ SR_{\text{Stock}}(X, \omega) = \left[ ARA + VRA \frac{(\Gamma - \beta)}{\beta} \right] \sigma_e - \frac{\alpha^2 \omega^2}{2 \beta^2 \sigma_e \sigma_X^2} \]

(x) The Sharpe Ratio on the option is given by

\[ SR_{\text{Option}}(X, \omega) = -\omega \beta \sigma_e^2 \left( 2 \Gamma^2 + 2 \Gamma (1 - \omega)^2 \beta + \beta \left[ 2 (1 - \omega)^2 \beta + \sigma_e^2 \right] \right) \sigma_X + \omega (\omega^2 \beta - 6 \sigma_e^2) \sigma_X^2 \]
\[ \frac{2 \beta^4 \sigma_e^2}{2 \beta^4 \sigma_e^2} \]
3.3.2 Analysis of portfolio strategies

The logarithmic investor is myopic and holds the mean-variance optimal portfolio. Hence by analyzing her optimal portfolio choice, we can understand the mean-variance component of the portfolio when financial markets are complete. The mean-variance portfolio weights for the stock and the derivative asset in the complete market economy are

\[
\theta_P^* = \frac{1}{\alpha R T} + \frac{\omega^2 (1 - \omega)}{\beta} \Gamma \alpha^3 + O\left(\alpha^4\right)
\]

\[
\theta_Q^* = -\frac{\omega \sigma_X}{\beta} \alpha - \frac{\alpha^2 \omega}{\beta^2} \left[ \Gamma + (1 - 2\omega) \beta \right] \sigma_X + O\left(\alpha^3\right)
\]

The myopic holding in the derivative asset when markets are complete is governed by the volatility of the drift of endowment growth rate, \(\sigma_X\). As \(\sigma_X\) increases, more of the derivative asset is held in the mean-variance portfolio. The availability of the derivative asset in the mean-variance portfolio also affects the optimal holding of the stock. As the conditional expected growth rate of the economy, \(\Gamma\), increases, less wealth is allocated to investment in the derivative and more to the stock.

The power utility investor is non-myopic and holds the mean-variance optimal portfolio (adjusted by risk tolerance) along with an intertemporal hedging portfolio, which allows the power utility investor to insure against changes in the stochastic investment opportunity set. The weights for the stock and the derivative asset in the hedging portfolio are

\[
H_P = -\frac{\omega (1 - \omega)}{\beta} \Gamma \alpha^3 + O\left(\alpha^4\right)
\]

\[
H_Q = \frac{\sigma_X}{\beta} \alpha + \frac{\alpha^2}{\beta^2} \left[ \Gamma + (1 - \omega) \beta \right] \sigma_X + O\left(\alpha^3\right)
\]

To leading order in \(\alpha\) the hedging portfolios for the stock and derivative are of opposite sign and both increase in magnitude as heterogeneity in risk aversion increases.

The proportion of wealth invested in the stock for hedging purposes increases as the distribution of wealth between the two agents in the economy becomes more even and the conditional expected growth rate of the economy increases.

As the volatility of the drift in the endowment growth rate increases the amount of wealth invested in the derivative for hedging purposes increases. An increase in the growth rate of the economy also increases the proportion of wealth invested in the derivative to insure against shifts in the investment opportunity set.
4 Comparing the economies without and with the derivative

We can now compare the economy with complete financial markets to that with incomplete markets. The following propositions summarize the differences across these two economies.

4.1 The effects on optimal portfolio choice

In the following proposition, we present the myopic component of the optimal portfolio and also the intertemporal hedging component for the economies with complete and incomplete markets.

**Proposition 3** To first order in $\alpha$, the mean-variance portfolio weights for the stock and the derivative asset in the complete market economy are

$$
\theta^*_P = \frac{1}{\alpha T} + \frac{\omega^2 (1 - \omega)}{\beta} \Gamma \alpha^3 + O \left( \alpha^4 \right)
$$

$$
\theta^*_Q = -\frac{\omega \sigma_X}{\beta} \alpha - \frac{\alpha^2 \omega}{\beta^2} \left[ \Gamma + (1 - 2\omega) \beta \right] \sigma_X + O \left( \alpha^3 \right)
$$

while in the incomplete market economy the mean variance portfolio weight for the stock is given by

$$
\theta^* = \frac{1}{\alpha T} - \frac{\omega^2 \sigma_X^2}{\beta^2 \sigma_e^2} \alpha^2 + O \left( \alpha^3 \right).
$$

The hedging portfolio that the power utility investor holds to insure against changes in the stochastic investment opportunity set in the complete market economy is

$$
H_P = -\frac{\omega (1 - \omega)}{\beta} \Gamma \alpha^3 + O \left( \alpha^4 \right)
$$

$$
H_Q = \frac{\sigma_X}{\beta} \alpha + \frac{\alpha^2 \omega}{\beta^2} \left[ \Gamma + (1 - \omega) \beta \right] \sigma_X + O \left( \alpha^3 \right)
$$

while in the incomplete market setting the hedging portfolio weight for the stock is given by

$$
H = \alpha^2 \frac{\omega \sigma_X^2}{\beta^2 \sigma_e^2} + O \left( \alpha^3 \right)
$$

Recall that the logarithmic investor is myopic and holds the mean-variance portfolio. Hence by analyzing her optimal portfolio choice, we can study how the mean-variance portfolio changes when markets are completed by the introducing the option. In the complete
market, the mean-variance portfolio contains holdings of the derivative and the underlying stock, whereas in the incomplete market there is no derivative asset.

The sizes of the hedging portfolio weights in the complete market economy are larger by an order of magnitude in $\alpha$ than in the incomplete market economy. Thus, it is clear that introducing a derivative to complete the market allows the power utility investor to hedge more fully.

### 4.2 The effects of on the cross-sectional wealth distribution

**Proposition 4** Denoting by superscript $c$ quantities for the economy where markets are complete and by $i$ quantities for the economy where markets are incomplete, we find that:

(i) The difference in the drift of the process for the distribution of wealth is

$$
\mu^c_\omega - \mu^i_\omega = \alpha \omega (1 - \omega) \left[ \sqrt{\sigma_e^2 + \frac{\sigma_X^2}{\beta^2}} - \sigma_e \right] - \alpha^2 \omega^2 (1 - \omega) \frac{\sigma_X^2}{\beta^2} + O(\alpha^3)
$$

(ii) The difference in the volatility of the process for the distribution of wealth is

$$
\sigma^c_\omega - \sigma^i_\omega = \alpha \omega (1 - \omega) \left[ \sqrt{\sigma_e^2 + \frac{\sigma_X^2}{\beta^2}} - \sigma_e \right] + O(\alpha^2)
$$

Regardless of whether there is any financial innovation that completes the market, the less risk averse investor will eventually dominate the economy by holding all the entire aggregate wealth. The rate at which this occurs is speeded up by financial innovation. The volatility of the cross-sectional wealth distribution is also effected by financial innovation. The wealth distribution of the least risk averse investor becomes more volatile, whilst that of the more risk averse investor becomes less volatile.

### 4.3 The effects on financial variables

We now consider the effect of introducing a derivative security on the riskless rate and the volatility of stock returns.

**Proposition 5** We find that:
(i) The difference in the volatility of stock returns is

$$\sigma^e - \sigma^i = \alpha^2 \omega (1 - \omega) \left[ \frac{\Gamma \omega + (1 - 3 \omega) \beta \sigma^2_e}{2 \beta^4 \sigma_e} \right] + O(\alpha^5)$$

(ii) The difference in the riskless rate is

$$r^e - r^i = \alpha^2 \omega (1 - \omega) \left[ \frac{\sigma^2_X}{\beta^2} + \frac{\Gamma}{\beta} \left( \sqrt{\sigma^2_e + \frac{\sigma^2_X}{\beta^2}} - \sigma_e \right) \right] + O(\alpha^3)$$

From the above expressions, we see that introducing the derivative leads to an increase in the volatility of stock returns as long as $\Gamma^2/2\sigma^2_e > \beta$. Also, the riskless interest increases unambiguously.

5 Conclusion

In this paper, we have evaluated the effect of introducing a contingent claim by comparing the equilibrium in an economy without and with this claim. The dynamic, general-equilibrium economy in which we have undertaken is this experiment is one where there are two investors, one with log utility and the other with power utility; the endowment process in this economy is assumed to be given by exogenous stochastic process and its growth rate is assumed to be stochastic as well. We find that the introduction of any non-redundant derivative increases the interest rate, and the volatility of stock returns. We also show how an investor would use this non-redundant derivative in her dynamic portfolio strategy.
A Derivations for the economy without the derivative

This appendix provides all the details of all the calculations required to obtain the results in the lemmas and propositions. In order to provide all the intermediate steps for obtaining the results, the appendix is not written in the typical “theorem” followed by “proof” style.

A.1 The Bellman equation

The Bellman equation for each investor is:

\[ 0 = \sup_{C, \theta} \left\{ f(C, J) + E \left( \frac{dJ(W, X)}{dt} \right) \right\} \]

where \( X = (X, \omega)' \). Now

\[ dJ = J_W dW + J_\omega d\omega + J_X dX + \frac{1}{2} J_{WW} (dW)^2 + \frac{1}{2} J_{\omega\omega} (d\omega)^2 + \frac{1}{2} J_{XX} (dX)^2 \]
\[ + J_W X dW dX + J_\omega X d\omega dX + J_W \omega dW d\omega \]

Defining \( c \equiv C/W \), we have that

\[ dW = W [r + \theta (\mu_R - r) - c] dt + W \theta \sigma P dZ_P \]
\[ dX = \mu X dt + \sigma X dZ_X \]
\[ d\omega = \mu \omega dt + \sigma \omega dZ_\omega \]

and that

\[ (dW)^2 = W^2 \theta^2 \sigma_P^2 dt \]
\[ (d\omega)^2 = \sigma_\omega^2 dt \]
\[ (dX)^2 = \sigma_X^2 dt \]
\[ dW.dX = W \theta \sigma P \sigma_X \rho_P X dt \]
\[ d\omega.dX = \sigma_X \sigma_\omega \rho_P X dt \]
\[ dW.d\omega = W \theta \sigma P \sigma_\omega dt \]

Thus,

\[ EdJ = E[J_W dW + J_\omega d\omega + J_X dX + \frac{1}{2} J_{WW} (dW)^2 + \frac{1}{2} J_{\omega\omega} (d\omega)^2 + \frac{1}{2} J_{XX} (dX)^2 \]
\[+ J_{WX} dW dX + J_{\omega X} d\omega dX + J_{W\omega} dW d\omega] \]

\[= \left[ (r + \theta (\mu - r) - c) dt \right] W J_W + J_{\omega\mu} dt \]

\[+ J_X \mu_X dt + \frac{1}{2} J_{WW} W^2 \theta^2 \sigma_\theta^2 dt + \frac{1}{2} J_{\omega\omega} \sigma_\theta^2 dt \]

\[+ \frac{1}{2} J_{XX} \sigma_X^2 dt + J_{WX} W \theta \sigma_P \sigma_X \rho_{PX} dt + J_{\omega X} \sigma_X \sigma_\omega \rho_{PX} dt \]

\[+ J_{W\omega} W \theta \sigma_P \sigma_\omega dt \]

Hence, the Bellman equation is:

\[0 = \sup_{C, \theta} \left\{ f(C, J) + (r + \theta (\mu - r) - c) W J_W + J_X \mu_X + J_{\omega\mu} \right. \]

\[+ \frac{1}{2} J_{WW} W^2 \theta^2 \sigma_\theta^2 + J_{WX} W \theta \sigma_P \sigma_X \rho_{PX} + J_{W\omega} W \theta \sigma_P \sigma_\omega \]

\[+ \frac{1}{2} J_{XX} \sigma_X^2 + J_{WX} \sigma_X \sigma_\omega \rho_{PX} + \frac{1}{2} J_{W\omega} \sigma_\theta^2 \right\} \]

Now to simplify the Bellman equation we start with

\[f(C, J) = \alpha^{-1} \beta [C^\alpha - \alpha J] \]

\[= \alpha^{-1} \beta C^\alpha - \alpha^{-1} \beta (e^gW)^\alpha \]

and that

\[J(W, X, \omega) = \frac{e^{\alpha g(X, \omega)} W^\alpha}{\alpha} \]

\[J_W = e^{\alpha g(X, \omega)} W^{\alpha - 1} \]

\[J_X = g_X e^{\alpha g(X, \omega)} W^\alpha \]

\[J_\omega = g_\omega e^{\alpha g(X, \omega)} W^\alpha \]

\[J_{WX} = \alpha g_X e^{\alpha g(X, \omega)} W^{\alpha - 1} \]

\[J_{W\omega} = \alpha g_\omega e^{\alpha g(X, \omega)} W^\alpha \]

\[J_{XX} = (g_{XX} + \alpha g_X g_\omega) e^{\alpha g(X, \omega)} W^\alpha \]

\[J_{W\omega} = (g_{\omega\omega} + \alpha g_\omega^2) e^{\alpha g(X, \omega)} W^\alpha \]

\[J_{WX} = \alpha g_X e^{\alpha g(X, \omega)} W^{\alpha - 1} \]

\[J_{WW} = (\alpha - 1) e^{\alpha g(X, \omega)} W^{\alpha - 2} \]
Note that
\[ \alpha^{-1} \beta C^\rho (e^g W)^{-\rho} = \alpha^{-1} c \]

Hence
\[
\begin{align*}
f(C, J) &= \alpha^{-1} c (e^g W)^\alpha - \alpha^{-1} \beta (e^g W)^\alpha \\
&= \alpha^{-1} (c - \beta) A (e^g W)^\alpha \\
&= \alpha^{-1} (c - \beta) (e^g W)^\alpha 
\end{align*}
\]

Note that
\[
(r + \theta (\mu R - r) - c) W J_W = (r + \theta (\mu R - r) - c) e^{\alpha g(X, \omega)} W^\alpha
\]

and that
\[
\begin{align*}
J_{X\mu_X} &= \mu_X g_X e^{\alpha g(X, \omega)} W^\alpha \\
J_{\omega\mu_\omega} &= \mu_\omega g_\omega e^{\alpha g(X, \omega)} W^\alpha \\
\frac{1}{2} J_{WW} W^2 \theta^2 \sigma_\theta^2 &\rho = \frac{1}{2} (\alpha - 1) \theta^2 \sigma_\theta^2 e^{\alpha g(X, \omega)} W^\alpha \\
J_{W\theta X}\sigma_{\rho X} &\rho = \alpha g_X \theta \sigma_{\rho X} \rho e^{\alpha g(X, \omega)} W^\alpha \\
J_{W\omega X}\sigma_{\rho X} &\rho = \alpha g_\theta \sigma_{\rho X} \rho e^{\alpha g(X, \omega)} W^\alpha \\
\frac{1}{2} J_{XX} \sigma_X^2 &\rho = \frac{1}{2} \sigma_X^2 \left( g_{XX} + \alpha g_X^2 \right) e^{\alpha g(X, \omega)} W^\alpha \\
J_{X\omega\sigma_{\rho X}} &\rho = \left( g_{X\omega} + \alpha g_X g_\omega \right) \sigma_{\rho X} \rho e^{\alpha g(X, \omega)} W^\alpha \\
\frac{1}{2} J_{\omega\omega} \sigma_\omega^2 &\rho = \frac{1}{2} \left( g_{\omega\omega} + \alpha g_\omega^2 \right) \sigma_\omega^2 e^{\alpha g(X, \omega)} W^\alpha 
\end{align*}
\]

Hence
\[
0 = \sup_{c, \theta} \left\{ \alpha^{-1} (c - \beta) + [r + \theta (\mu R - r) - c] + \frac{1}{2} (\alpha - 1) \theta^2 \sigma_\theta^2 + \mu_X g_X + \mu_\omega g_\omega \\
+ \alpha g_X \theta \sigma_{\rho X} \rho \rho + \alpha g_\omega \theta \sigma_{\rho \omega} \\
+ \frac{1}{2} \sigma_X^2 \left( g_{XX} + \alpha g_X^2 \right) + (g_{X\omega} + \alpha g_X g_\omega) \sigma_{X\omega} \rho X + \frac{1}{2} \left( g_{\omega\omega} + \alpha g_\omega^2 \right) \sigma_\omega^2 \right\} \\
= \sup_{c, \theta} \left\{ (1 - \alpha) c - \beta + \alpha [r + \theta (\mu R - r)] + \frac{1}{2} \alpha (\alpha - 1) \theta^2 \sigma_\theta^2 + \alpha g_X g_X + \alpha g_\omega g_\omega \\
+ \alpha^2 g_X \theta \sigma_{\rho X} \rho X + \alpha^2 g_\omega \theta \sigma_{\rho \omega} \\
+ \frac{1}{2} \alpha \sigma_X^2 \left( g_{XX} + \alpha g_X^2 \right) + \alpha (g_{X\omega} + \alpha g_X g_\omega) \sigma_{X\omega} \rho X + \frac{1}{2} \alpha \left( g_{\omega\omega} + \alpha g_\omega^2 \right) \sigma_\omega^2 \right\}
\]
A.2 The equilibrium

A.2.1 Consumption policies

The first order condition for consumption is

\[ \frac{\partial f(C, J)}{\partial C} = J_W. \]

Thus, the optimal consumption-wealth ratio of the power utility investor is

\[ c = \left( \beta^{-1} \exp \alpha g \right)^{\frac{1}{\alpha-1}} \]

and for the logarithmic investor it is

\[ c^* = \beta. \]

A.2.2 Portfolio policies

We know that,

\[ \theta^* = \frac{\mu_R - r}{\sigma_P^2} \]

\[ \theta = \frac{1}{1 - \alpha} \left( \frac{\mu_R - r}{\sigma_P^2} + \alpha \sigma_P^{-1} \sigma_X \rho_{PXgX} \right) . \]

Hence,

\[ \theta = \frac{1}{1 - \alpha} \left( \theta^* + \alpha \sigma_P^{-1} \sigma_X \rho_{PXgX} \right) . \]

Using the market clearing condition

\[ \theta \omega + \theta^* (1 - \omega) \]

one can show that

\[ \theta = \frac{1}{1 - \alpha (1 - \omega)} + \frac{\alpha (1 - \omega)}{1 - \alpha (1 - \omega)} \sigma_P^{-1} \sigma_X \rho_{PXgX} \]

and

\[ \theta^* = \frac{1 - \alpha}{1 - \alpha (1 - \omega)} - \frac{\alpha \omega}{1 - \alpha (1 - \omega)} \sigma_P^{-1} \sigma_X \rho_{PXgX} . \]
A.2.3 Interest rate

Recalling that

$$\theta^* = \frac{\mu_R - r}{\sigma_R^2}$$

we see that

$$r = \mu_R - \sigma_R^2 \theta^*.$$  

A.2.4 Cross-sectional distribution of wealth

Now we derive the cross-sectional wealth distribution. Define the state variable \( \omega \equiv \frac{W}{W^* + W} \). This state variable summarizes the cross-sectional wealth distribution of the economy. In equilibrium \( \omega \) and \( X \) completely characterize the investment opportunity set faced by the agents

$$\omega = \frac{W}{W^* + W} = W(W^* + W)^{-1} \Rightarrow$$

$$d\omega = \omega_{W^*} dW^* + \omega_W dW + \frac{1}{2} \omega_{W^*} W dW^* dW + \omega_W W dW^* dW + \frac{1}{2} \omega_{WW} dW dW$$

where

$$\omega_{W^*} = -W(W^* + W)^{-2}$$

$$\omega_W = (W^* + W)^{-1} - W(W^* + W)^{-2}$$

$$\omega_{W^*W^*} = 2W(W^* + W)^{-3}$$

$$\omega_{WW} = -2(W^* + W)^{-2} + 2W(W^* + W)^{-3}$$

$$\omega_{WW} = -(W^* + W)^{-2} + 2W(W^* + W)^{-3}$$

Hence

$$d\omega = -W(W^* + W)^{-2} dW^* + [(W^* + W)^{-1} - W(W^* + W)^{-2}] dW + W(W^* + W)^{-3} dW^* dW^* + \left[ - (W^* + W)^{-2} + 2W(W^* + W)^{-3} \right] dW^* dW$$

$$+ \left[ -(W^* + W)^{-2} + W(W^* + W)^{-3} \right] dW \cdot dW$$

$$= -W(W^* + W)^{-2} \left[ \{W^* [r + \theta^* (\mu_R - r) - c^*] \right] dt + W^* \theta^* \sigma_P dZ_P \right]$$

$$+ \left[ (W^* + W)^{-1} - W(W^* + W)^{-2} \right] \left[ \{W [r + \theta (\mu_R - r) - c] \right] dt + W \theta \sigma_P dZ_P \right] + \ldots$$
\[
W(W^* + W)^{-3} W^* \theta^2 \sigma_P^2 dt + \left[- (W^* + W)^{-2} + 2W(W^* + W)^{-3}\right] W^* W \theta^* \theta \sigma_P^2 dt \\
+ \left[- (W^* + W)^{-2} + W(W^* + W)^{-3}\right] W^2 \theta^2 \sigma_P^2 dt
\]

\[
= \{-W(W^* + W)^{-2} \{W^*[r + \theta^* (\mu_R - r) - c^*] \}
+ W(W^* + W)^{-1}\left[1 - W(W^* + W)^{-1}\right] [r + \theta(\mu_R - r) - c]
+ W(W^* + W)^{-3} W^* \theta^2 \sigma_P^2 + \left[- (W^* + W)^{-2} + 2W(W^* + W)^{-3}\right] W^* W \theta^* \theta \sigma_P^2 \\
+ \left[- (W^* + W)^{-2} + W(W^* + W)^{-3}\right] W^2 \theta^2 \sigma_P^2 \} dt + \\
-W(W^* + W)^{-2} W^* \theta^* \sigma_P dZ_P + \left[(W^* + W)^{-1} - W(W^* + W)^{-2}\right] W \theta \sigma_P dZ_P
\]

\[
= \{-\omega(1 - \omega) \{r + \theta^* (\mu_R - r) - c^*\} + \omega(1 - \omega) [r + \theta(\mu_R - r) - c]
+ (1 - \omega)^2 \theta^2 \sigma_P^2 + \left[2\omega^2 (1 - \omega) - \omega(1 - \omega)\right] \theta^* \theta \sigma_P^2 - \omega^2 (1 - \omega) \theta^2 \sigma_P^2 \} dt \\
+ \{-\omega(1 - \omega) \theta \sigma_P + \omega(1 - \omega) \theta \sigma_P\} dZ_P \\
= \omega(1 - \omega) \left\{\{-\theta^* (\mu_R - r) + c^* + \theta(\mu_R - r) - c\} \right. \\
+ (1 - \omega) \theta^2 \sigma_P^2 + \left[2\omega - 1\right] \theta^* \theta \sigma_P^2 - \omega \theta^2 \sigma_P^2 \} dt \\
+ \{-\theta^* \sigma_P + \theta \sigma_P\} dZ_P \}
\]

\[
\theta \omega(1 - \omega) \left\{\left[\mu_R - r\right] (\theta - \theta^*) - c + c^* - \sigma_P^2 \left(\theta - \theta^*\right) [\omega\theta + (1 - \omega) \theta^*] \right. \\
+ \left. \sigma_P (\theta - \theta^*) dZ_P \right\} .
\]

Therefore

\[
d\omega = \omega(1 - \omega) \left\{\left[\mu_R - r\right] (\theta - \theta^*) - c + c^* - \sigma_P^2 \left(\theta - \theta^*\right) [\omega\theta + (1 - \omega) \theta^*] \right. \\
+ \left. \sigma_P (\theta - \theta^*) dZ_P \right\}.
\]

Now

\[
\theta \omega + (1 - \omega) \theta^* = \frac{W \theta + W^* \theta^*}{W + W^*} = 1.
\]

Therefore

\[
d\omega = \omega(1 - \omega) \left\{\left[\mu_R - r\right] (\theta - \theta^*) - c + c^* - \sigma_P^2 \left(\theta - \theta^*\right) \right. \\
+ \left. \sigma_P (\theta - \theta^*) dZ_P \right\}
\]

Hence

\[
d\omega = \mu_\omega dt + \sigma_\omega dZ_P,
\]

Non-Redundant Derivatives
where
\[
\begin{align*}
\mu_\omega &= \omega (1 - \omega) \left( (\mu_R - r - \sigma_P^2) (\theta - \theta^*) - c + c^* \right) \\
\sigma_\omega &= \omega (1 - \omega) \sigma_P (\theta - \theta^*)
\end{align*}
\]

### A.2.5 Stock price and price-dividend ratio

Market clearing in the consumption good gives
\[
\frac{e}{p} = \omega c + (1 - \omega) c^*.
\]

Thus the stock price is given by
\[
P = p.e
\]
where
\[
p = [\omega c + (1 - \omega) c^*]^{-1}.
\]

Therefore,
\[
\frac{dP}{P} = \frac{de}{e} + \frac{dp}{p} + \frac{dp.de}{p.e}.
\]

We know that
\[
\frac{de}{e} = \mu_e dt + \sigma_e dZ_e,
\]
and
\[
\frac{dp}{p} = \frac{p_\omega}{p} d\omega + \frac{p_X}{p} dX + \frac{1}{2} \frac{p_{XX}}{p} dX.dX + \frac{p_\omega X}{p} d\omega.dX + \frac{p_{XX}}{p} d\omega.d\omega.
\]

Given that there is only a single risky asset, \(dZ_\omega = dZ_P\). Hence,
\[
\frac{dP}{P} = \left[ \mu_e + \sigma_e \left( \frac{p_\omega}{p} \rho_{e\omega} + \frac{p_X}{p} \rho_{eX} \right) + \mu_\omega \frac{p_\omega}{p} + \mu_X \frac{p_X}{p} \\
+ \frac{1}{2} \sigma_X^2 \frac{p_{XX}}{p} + \rho_{X\omega} \sigma_\omega \sigma_X \frac{p_{X\omega}}{p} + \frac{1}{2} \sigma_\omega^2 \frac{p_{XX}}{p} \right] dt \\
+ \sigma_\omega \frac{p_\omega}{p} dZ_P + \sigma_X \frac{p_X}{p} dZ_X + \sigma_e dZ_e
\]
Now

\[ \frac{dP + edt}{P} = \frac{dP}{P} + p^{-1}dt. \]

Therefore,

\[ \frac{dP + edt}{P} = \left[ p^{-1} + \mu_e + \sigma_e \left( \sigma_\omega \frac{p_e}{p} \rho_{e\omega} + \sigma_X \frac{p_X}{p} \rho_{eX} \right) + \mu_\omega \frac{p_\omega}{p} + \mu_X \frac{p_X}{p} \right. \]

\[ + \frac{1}{2} \sigma_X^2 \frac{p_{XX}}{p} + \rho_{\omega X} \sigma_\omega \sigma_X \frac{p_{\omega X}}{p} + \frac{1}{2} \sigma_\omega^2 \frac{p_{\omega\omega}}{p} \right] dt \]

\[ + \sigma_\omega \frac{p_\omega}{p} dZ_P + \sigma_X \frac{p_X}{p} dZ_X + \sigma_e dZ_e. \]

Hence,

\[ \frac{dP + edt}{P} = \mu_R dt + \sigma_p dZ_P \]

where

\[ \mu_R = p^{-1} + \mu_e + \sigma_e \left( \sigma_\omega \frac{p_e}{p} \rho_{e\omega} + \sigma_X \frac{p_X}{p} \rho_{eX} \right) + \mu_\omega \frac{p_\omega}{p} + \mu_X \frac{p_X}{p} \]

\[ + \frac{1}{2} \sigma_X^2 \frac{p_{XX}}{p} + \rho_{\omega X} \sigma_\omega \sigma_X \frac{p_{\omega X}}{p} + \frac{1}{2} \sigma_\omega^2 \frac{p_{\omega\omega}}{p}, \]

and

\[ \sigma_p dZ_P = \sigma_\omega \frac{p_\omega}{p} dZ_P + \sigma_X \frac{p_X}{p} dZ_X + \sigma_e dZ_e. \]

We can rearrange the latter expression to obtain

\[ \left( 1 - \omega (1 - \omega) \left( \theta_t - \theta_t^* \right) \frac{p_\omega}{p} \right) \sigma_p dZ_P = \sigma_X \frac{p_X}{p} dZ_X + \sigma_e dZ_e. \]

Hence

\[ \sigma_p = \left( 1 - \omega (1 - \omega) \left( \theta_t - \theta_t^* \right) \frac{p_\omega}{p} \right)^{-1} \left[ \left( \sigma_X \frac{p_X}{p} \right)^2 + 2 \rho_{eX} \sigma_e \sigma_X \frac{p_X}{p} + \sigma_e^2 \right]^{1/2}, \]

and

\[ dZ_P = \sigma_p^{-1} \left( 1 - \omega (1 - \omega) \left( \theta_t - \theta_t^* \right) \frac{p_\omega}{p} \right)^{-1} \left( \sigma_X \frac{p_X}{p} dZ_X + \sigma_e dZ_e \right). \]

We can now find expressions for the correlations \( \rho_{p_e} \) and \( \rho_{p_X} \):

\[ \rho_{p_e} = \sigma_p^{-1} \left( 1 - \omega (1 - \omega) \left( \theta_t - \theta_t^* \right) \frac{p_\omega}{p} \right)^{-1} \left( \rho_{eX} \sigma_X \frac{p_X}{p} + \sigma_e \right). \]

\[ \rho_{p_X} = \sigma_p^{-1} \left( 1 - \omega (1 - \omega) \left( \theta_t - \theta_t^* \right) \frac{p_\omega}{p} \right)^{-1} \left( \sigma_X \frac{p_X}{p} \rho_{eX} \sigma_e \right). \]
It is straightforward to obtain the zero-th order expansions in $\alpha$ for the optimal policies and prices because in this case the problem reduces to the case where all agents have log utility. Using these zero order expressions we can find first order expressions for the equilibrium variables and also get a partial differential equation for $g_1$ by expanding the Bellman equation in $\alpha$. We can solve this partial differential equation to find $g_1$. We can repeat this procedure as follows: From the $(n - 1)$ order expressions we compute the equilibrium variables to order $n$ and obtain a partial differential equation for $g_n$. If we can solve this partial differential equation we can compute the $(n + 1)$ order equilibrium variables and find a partial differential equation for $g_{n+1}$. Using this method we can find closed form expressions for the equilibrium variables and $g$ to fourth order in $\alpha$. 
B Derivations for economy with non-redundant derivative

B.1 The Bellman equation and first-order conditions

Consider the power utility investor. This investor has the dynamic budget constraint

\[
\frac{dW}{W} = \left[r + \theta_P \left(\mu_P + \frac{e}{D} - r\right) + \theta_Q (\mu_Q - r) - c\right] dt + \theta_P \sigma_P dZ_P + \theta_Q \sigma_Q dZ_Q.
\]

Then, the Bellman equation is

\[
0 = \sup_{C,\theta} \left\{ f(C, J) + \mathbb{E} \left( \frac{dJ(W, X)}{dt} \right) \right\}
\]

where \( X = (X, \omega)' \). Now

\[
dJ = J_W dW + J_\omega d\omega + J_X dX + \frac{1}{2} J_{WW} (dW)^2 + \frac{1}{2} J_{\omega\omega} (d\omega)^2 + \frac{1}{2} J_{XX} (dX)^2 + J_{WX} dW dX + J_{\omega X} d\omega dX + J_{W\omega} dW d\omega.
\]

so that the Bellman equation can be written as:

\[
0 = \sup_{C,\theta_P,\theta_Q} \left\{ f(C, J) + (r + \theta_P (\mu_R - r) + \theta_Q (\mu_Q - r) - c) W J_W + \mu_X J_X
\right.
\]

\[
+ \frac{1}{2} \left( \theta_P^2 \sigma_P^2 + 2 \rho_{PQ} \theta_P \theta_Q \sigma_P \sigma_Q + \theta_Q^2 \sigma_Q^2 \right) W^2 J_{WW}
\]

\[
+ \left( \theta_P \sigma_P \rho_P X + \theta_Q \sigma_Q \rho_Q X \right) \sigma_X W J_{WX} + \frac{1}{2} \sigma_X J_{XX} \sigma_X \right\}.
\]

Note that

\[
J(X, \omega) = \frac{W^\alpha e^{\alpha g}}{\alpha}
\]

Hence the Bellman equation becomes

\[
0 = \left[r + \theta_P (\mu_R - r) + \theta_Q (\mu_Q - r) - \beta \frac{1}{1-\alpha} e^{\frac{\alpha g}{2-\alpha}}\right] + \mu_\omega g_\omega + \mu_X g_X
\]

\[
+ \frac{1}{2} (\alpha - 1) \left( \theta_P^2 \sigma_P^2 + 2 \rho_{PQ} \theta_P \theta_Q \sigma_P \sigma_Q + \theta_Q^2 \sigma_Q^2 \right)
\]

\[
+ \frac{1}{2} \sigma^2 \left(g_{\omega\omega} + \alpha g_\omega^2 \right) + \frac{1}{2} \sigma_X^2 \left(g_{XX} + \alpha g_X^2 \right)
\]

\[
+ \alpha \left( \theta_P \sigma_P \rho_P X + \theta_Q \sigma_Q \rho_Q X \right) \sigma_X g_X + \sigma_\omega \sigma_X \rho_\omega X (g_{\omega X} + \alpha g_\omega g_X)
\]

\[
+ \alpha \left( \theta_P \sigma_P \rho_P \omega + \theta_Q \sigma_Q \rho_Q \omega \right) \sigma_\omega g_\omega
\]

\[
+ \alpha^{-1} \left( \beta \frac{1}{1-\alpha} e^{\frac{\alpha g}{2-\alpha}} - \beta \right),
\]
which simplifies to

\[
\beta = (1 - \alpha) \beta^{\frac{1}{\alpha}} e^{-\frac{\alpha}{\alpha-1}} + \alpha \left[ r + \theta_P (\mu_R - r) + \theta_Q (\mu_Q - r) \right] \\
+ \frac{1}{2} \alpha (\alpha - 1) \left( \theta_P^2 \sigma_P^2 + 2 \rho_{PQ} \theta_P \theta_Q \sigma_P \sigma_Q + \theta_P^2 \sigma_Q^2 \right) \\
+ \frac{1}{2} \alpha \sigma_Z^2 \left( g_{\omega} + \alpha g_{\omega} \right) + \frac{1}{2} \alpha \sigma_X^2 \left( g_{XX} + \alpha g_{XX} \right) \\
+ \alpha^2 (\theta_P \sigma_P \rho_{PX} + \theta_Q \sigma_Q \rho_{QX}) \sigma_X g_X \\
+ \alpha \sigma_X \rho_{X} \sigma_X \left( g_{\omega X} + \alpha g_{\omega X} \right) \\
+ \alpha^2 (\theta_P \sigma_P \rho_{P} + \theta_Q \sigma_Q \rho_{Q}) \sigma_X g_X.
\]

The first order conditions are

\[
\frac{\partial f(C, J)}{\partial C} = J_W \\
0 = (\mu_R - r) W J_W + \left( \theta_P \sigma_P^2 + \rho_{PQ} \theta_Q \sigma_P \sigma_Q \right) W^2 J_{WW} + \sigma_P \rho_{P} \sigma_X \sigma_W J_{WX} \\
0 = (\mu_Q - r) W J_W + \left( \theta_Q \sigma_Q^2 + \rho_{PQ} \theta_P \sigma_P \sigma_Q \right) W^2 J_{WW} + \sigma_Q \rho_{Q} \sigma_X \sigma_W J_{WX}
\]

Define

\[
\phi_P = \frac{\mu_R - r}{\sigma_P} \\
\phi_Q = \frac{\mu_Q - r}{\sigma_Q}
\]

Then

\[
0 = \sigma_P \phi_P W J_W + \left( \theta_P \sigma_P^2 + \rho_{PQ} \theta_Q \sigma_P \sigma_Q \right) W^2 J_{WW} + \sigma_P \rho_{P} \sigma_X \sigma_W J_{WX} \\
0 = \sigma_Q \phi_Q W J_W + \left( \theta_Q \sigma_Q^2 + \rho_{PQ} \theta_P \sigma_P \sigma_Q \right) W^2 J_{WW} + \sigma_Q \rho_{Q} \sigma_X \sigma_W J_{WX}
\]

In matrix form this is

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
\sigma_P \phi_P W J_W + \sigma_P \rho_{P} \sigma_X \sigma_W J_{WX} \\
\sigma_Q \phi_Q W J_W + \sigma_Q \rho_{Q} \sigma_X \sigma_W J_{WX}
\end{bmatrix} + W^2 J_{WW} \begin{bmatrix}
\sigma_P^2 \\
\rho_{PQ} \sigma_P \sigma_Q \sigma_Q^2
\end{bmatrix} \begin{bmatrix}
\theta_P \\
\theta_Q
\end{bmatrix}
\]

Hence

\[
\begin{bmatrix}
\theta_P \\
\theta_Q
\end{bmatrix} = - \left( W^2 J_{WW} \right)^{-1} \begin{bmatrix}
\sigma_P^2 \\
\rho_{PQ} \sigma_P \sigma_Q \sigma_Q^2
\end{bmatrix}^{-1} \begin{bmatrix}
\sigma_P \phi_P W J_W + \sigma_P \rho_{P} \sigma_X \sigma_W J_{WX} \\
\sigma_Q \phi_Q W J_W + \sigma_Q \rho_{Q} \sigma_X \sigma_W J_{WX}
\end{bmatrix}
\]
\[ \begin{aligned}
&= - \left( W^2 J_{WW} \right)^{-1} \left[ \sigma_P^2 \sigma_Q^2 \left( 1 - \rho_{PQ}^2 \right) \right]^{-1} \\
&\quad \begin{bmatrix}
\sigma_Q^2 \\
-\rho_{PQ} \sigma_P \sigma_Q
\end{bmatrix}
\begin{bmatrix}
\sigma_P (\phi_P W J_W + \rho_P X \sigma_X W J_W X) \\
\sigma_Q (\phi_Q W J_W + \rho_Q X \sigma_X W J_W X)
\end{bmatrix}
\begin{bmatrix}
\sigma_P^2 \\
-\rho_{PQ} \sigma_P \sigma_Q
\end{bmatrix}
\begin{bmatrix}
\sigma_P (\phi_P + \alpha \rho_P X \sigma_X g'(X)) \\
\sigma_Q (\phi_Q + \alpha \rho_Q X \sigma_X g'(X))
\end{bmatrix}
\end{aligned} \\
&= (1 - \alpha)^{-1} \left[ \sigma_P^2 \sigma_Q^2 \left( 1 - \rho_{PQ}^2 \right) \right]^{-1} \\
&\quad \begin{bmatrix}
\sigma_Q^2 \\
-\rho_{PQ} \sigma_P \sigma_Q
\end{bmatrix}
\begin{bmatrix}
\sigma_P (\phi_P - \rho_{PQ} \phi_Q + \alpha (\rho_P X - \rho_{PQ} \rho_Q X) \sigma_X g'(X)) \\
\sigma_Q (\phi_Q - \rho_{PQ} \phi_P)
\end{bmatrix}
\begin{bmatrix}
\sigma_P^2 \\
-\rho_{PQ} \sigma_P \sigma_Q
\end{bmatrix}
\begin{bmatrix}
\sigma_Q (\phi_Q + \alpha \rho_Q X \sigma_X g'(X))
\end{bmatrix}
\end{aligned} \\
\]

Therefore, the optimal policy rules of the investor with power utility are

\[
\begin{aligned}
\theta_P &= (1 - \alpha)^{-1} \left[ \sigma_P \left( 1 - \rho_{PQ}^2 \right) \right]^{-1} \left[ \phi_P - \rho_{PQ} \phi_Q + \alpha (\rho_P X - \rho_{PQ} \rho_Q X) \sigma_X g'(X) \right] \\
\theta_Q &= (1 - \alpha)^{-1} \left[ \sigma_Q \left( 1 - \rho_{PQ}^2 \right) \right]^{-1} \left[ \phi_Q - \rho_{PQ} \phi_P + \alpha (\rho_Q X - \rho_{PQ} \rho_P X) \sigma_X g'(X) \right]
\end{aligned}
\]

And, the optimal policy rules for the investor with logarithmic utility are

\[
\begin{aligned}
\theta_P^* &= \left[ \sigma_P \left( 1 - \rho_{PQ}^2 \right) \right]^{-1} \left[ \phi_P - \rho_{PQ} \phi_Q \right] \\
\theta_Q^* &= \left[ \sigma_Q \left( 1 - \rho_{PQ}^2 \right) \right]^{-1} \left[ \phi_Q - \rho_{PQ} \phi_P \right]
\end{aligned}
\]

**B.2 The equilibrium**

**B.2.1 Consumption policies**

From the first order condition for consumption, the optimal consumption-wealth ratio of the power utility investor is

\[
c = \left( \beta^{-1} \exp \alpha g \right)^\frac{1}{\alpha+1}
\]

and for the logarithmic investor it is

\[
c^* = \beta.
\]

**B.2.2 Portfolio policies**

We know that the optimal portfolio policies are given by

\[
\begin{aligned}
\theta_P &= (1 - \alpha)^{-1} \left[ \sigma_P \left( 1 - \rho_{PQ}^2 \right) \right]^{-1} \left[ \phi_P - \rho_{PQ} \phi_Q + \alpha (\rho_P X - \rho_{PQ} \rho_Q X) \sigma_X g'(X) \right] \\
\theta_Q &= (1 - \alpha)^{-1} \left[ \sigma_Q \left( 1 - \rho_{PQ}^2 \right) \right]^{-1} \left[ \phi_Q - \rho_{PQ} \phi_P + \alpha (\rho_Q X - \rho_{PQ} \rho_P X) \sigma_X g'(X) \right]
\end{aligned}
\]
θ_P^* = \left[ \sigma_P \left(1 - \rho_{PQ}^2\right) \right]^{-1} \left[ \phi_P - \rho_{PQ} \phi_Q \right]
θ_Q^* = \left[ \sigma_Q \left(1 - \rho_{PQ}^2\right) \right]^{-1} \left[ \phi_Q - \rho_{PQ} \phi_P \right].

We also have the market clearing conditions:

\theta_P \omega + \theta_P^* (1 - \omega) = 1;
\theta_Q \omega + \theta_Q^* (1 - \omega) = 0.

Hence we can show that

θ_P = \frac{1}{1 - \alpha (1 - \omega)} + \frac{\alpha (1 - \omega)}{1 - \alpha (1 - \omega)} \left[ \sigma_P \left(1 - \rho_{PQ}^2\right) \right]^{-1} (\rho_{PX} - \rho_{PQ} \rho_{QX} \sigma_{XgX})
θ_P^* = \frac{1 - \alpha}{1 - \alpha (1 - \omega)} - \frac{\alpha \omega}{1 - \alpha (1 - \omega)} \left[ \sigma_P \left(1 - \rho_{PQ}^2\right) \right]^{-1} (\rho_{PX} - \rho_{PQ} \rho_{QX} \sigma_{XgX})
θ_Q = \frac{\alpha (1 - \omega)}{1 - \alpha (1 - \omega)} \left[ \sigma_Q \left(1 - \rho_{PQ}^2\right) \right]^{-1} (\rho_{QX} - \rho_{PQ} \rho_{PX} \sigma_{XgX})
θ_Q^* = -\frac{\alpha \omega}{1 - \alpha (1 - \omega)} \left[ \sigma_Q \left(1 - \rho_{PQ}^2\right) \right]^{-1} (\rho_{QX} - \rho_{PQ} \rho_{PX} \sigma_{XgX}).

### B.2.3 Interest rate

From

θ_Q = (1 - \alpha)^{-1} \left[ \sigma_Q \left(1 - \rho_{PQ}^2\right) \right]^{-1} \left[ \phi_Q - \rho_{PQ} \phi_P + \alpha (\rho_{QX} - \rho_{PQ} \rho_{PX}) \sigma_{Xg}'(X) \right]
θ_Q^* = \left[ \sigma_Q \left(1 - \rho_{PQ}^2\right) \right]^{-1} (\phi_Q - \rho_{PQ} \phi_P)

and

0 = \theta_Q \omega + \theta_Q^* (1 - \omega)

we can show that

r = \frac{\mu_Q \sigma_P - \rho_{PQ} \sigma_{QsR}}{\sigma_P - \rho_{PQ} \sigma_Q} + \frac{\alpha \omega}{1 - \alpha (1 - \omega)} \frac{\sigma_P \sigma_Q}{\sigma_P - \rho_{PQ} \sigma_Q} (\rho_{QX} - \rho_{PQ} \rho_{PX} \sigma_{Xg})

Alternatively, from

θ_P = (1 - \alpha)^{-1} \left[ \sigma_P \left(1 - \rho_{PQ}^2\right) \right]^{-1} \left[ \phi_P - \rho_{PQ} \phi_Q + \alpha (\rho_{PX} - \rho_{PQ} \rho_{QX}) \sigma_{Xg}'(X) \right]
θ_P^* = \left[ \sigma_P \left(1 - \rho_{PQ}^2\right) \right]^{-1} (\phi_P - \rho_{PQ} \phi_Q),

1 = \theta_P \omega + \theta_P^* (1 - \omega),
we can show that:

\[ r = \frac{\mu_R \sigma_Q - \rho_{PQ} \sigma_P \mu_Q}{\sigma_Q - \rho_{PQ} \sigma_Q} - \frac{1 - \alpha}{1 - \alpha (1 - \omega)} \frac{\sigma_P \sigma_Q \left(1 - \rho_{PQ}^2\right)}{\sigma_Q - \rho_{PQ} \sigma_P} \]

\[ + \frac{\alpha (1 - \omega)}{1 - \alpha (1 - \omega)} \frac{\sigma_P \sigma_Q}{(\sigma_Q - \rho_{PQ} \sigma_P)} \left(\rho_{PQ} X - \rho_{PQ} \rho_Q X\right) \sigma_X g'(X) \]

\[ = \frac{\mu_R \sigma_Q - \rho_{PQ} \sigma_P \mu_Q}{\sigma_Q - \rho_{PQ} \sigma_Q} - \theta_p' \frac{\sigma_P \sigma_Q \left(1 - \rho_{PQ}^2\right)}{\sigma_Q - \rho_{PQ} \sigma_P}. \]

### B.2.4 Cross-sectional distribution of wealth

Now consider the state variable \( \omega \)

\[ \omega = \frac{W}{W^* + W} = W (W^* + W)^{-1} \Rightarrow \]

\[ d\omega = \omega W^* dW^* + \omega_W dW + \frac{1}{2} \omega W^* dW^* dW^* + \omega_W W dW^* dW + \frac{1}{2} \omega W W dW dW \]

where

\[ \omega_{W^*} = -W (W^* + W)^{-2} \]

\[ \omega_W = (W^* + W)^{-1} - W (W^* + W)^{-2} \]

\[ \omega_{W^*W^*} = 2W (W^* + W)^{-3} \]

\[ \omega_{WW} = -2 (W^* + W)^{-2} + 2W (W^* + W)^{-3} \]

\[ \omega_{W^*W} = - (W^* + W)^{-2} + 2W (W^* + W)^{-3} \]

\[ d\omega = \omega_W dW^* + \omega_W dW + \frac{1}{2} \omega W^* W dW^* dW^* + \omega W W dW^* dW + \frac{1}{2} \omega W W dW dW \]

We have the wealth equation for the logarithmic agent

\[ dW^* = W^* \left[ r + \theta_P (\mu_R - r) + \theta_Q (\mu_Q - r) - c^* \right] dt + W^* \left( \theta_P \sigma_P dZ_P + \theta_Q \sigma_Q dZ_Q \right) \]

and for the agent with power utility

\[ dW = W \left[ r + \theta_P (\mu_R - r) + \theta_Q (\mu_Q - r) - c \right] dt + W \left( \theta_P \sigma_P dZ_P + \theta_Q \sigma_Q dZ_Q \right) \]

Hence

\[ dW. dW = W^2 \left( \theta_P^2 \sigma_P^2 + 2 \rho_{PQ} \sigma_P \sigma_Q \theta_P \theta_Q + \theta_Q^2 \sigma_Q^2 \right) dt \]

\[ dW^* dW^* = W^2 \left( \theta_P^2 \sigma_P^2 + 2 \rho_{PQ} \sigma_P \sigma_Q \theta_P \theta_Q + \theta_Q^2 \sigma_Q^2 \right) dt \]

\[ dW^* dW = W^* W \left[ \theta_P^* \theta_P \sigma_P^2 + \rho_{eX} \left( \theta_P \theta_Q^* + \theta_P^* \theta_Q \right) \sigma_P \sigma_Q + \theta_Q^* \theta_Q \sigma_Q^2 \right] dt \]
so that

\[
d\omega = -\frac{WW^*}{(W + W^*)^2} \left[ \left( r + \theta_P^* (\mu_R - r) + \theta_Q^* (\mu_Q - r) - c^* \right) dt + \theta_P \sigma_P dZ_P + \theta_Q^* \sigma_Q dZ_Q \right] \\
+ \left( \frac{W}{W + W^*} - \frac{W^2}{(W + W^*)^2} \right) \left( [r + \theta_P (\mu_R - r) + \theta_Q (\mu_Q - r) - c] dt + \theta_P \sigma_P dZ_P + \theta_Q \sigma_Q dZ_Q \right) \\
+ \frac{WW^*}{(W + W^*)^3} \left( \theta_P^2 \sigma_P^2 + 2\rho_P \sigma_P \sigma_Q \theta_P^* \theta_Q^* + \theta_Q^2 \sigma_Q^2 \right) dt \\
+ \left( \frac{2W^2W^*}{(W + W^*)^3} - \frac{WW^*}{(W + W^*)^2} \right) \left[ \theta_P \theta_P^* \sigma_P^2 + \left( \theta_P \theta_Q^* + \theta_P^* \theta_Q \right) \rho_P \sigma_P \sigma_Q + \theta_Q \theta_Q^* \sigma_Q^2 \right] dt \\
+ \left( \frac{W^3}{(W + W^*)^3} - \frac{W^2}{(W + W^*)^2} \right) \left( \theta_P^2 \sigma_P^2 + 2\rho_P \sigma_P \sigma_Q \theta_P \theta_Q + \theta_Q^2 \sigma_Q^2 \right) dt \\
= -\omega (1 - \omega) \left( r + \theta_P^* (\mu_R - r) + \theta_Q^* (\mu_Q - r) - c^* \right) dt \\
- \omega (1 - \omega) \left( \theta_P^* \sigma_P dZ_P + \theta_Q^* \sigma_Q dZ_Q \right) \\
+ \omega (1 - \omega) (r + \theta_P (\mu_R - r) + \theta_Q (\mu_Q - r) - c) \\
+ \omega (1 - \omega) (\theta_P \sigma_P dZ_P + \theta_Q \sigma_Q dZ_Q) \\
+ \omega (1 - \omega)^2 \left( \theta_P^2 \sigma_P^2 + 2\rho_P \sigma_P \sigma_Q \theta_P^* \theta_Q^* + \theta_Q^2 \sigma_Q^2 \right) dt \\
+ \left( 2\omega^2 (1 - \omega) - \omega (1 - \omega) \right) \left( \theta_P \theta_P^* \sigma_P^2 + \left( \theta_P \theta_Q^* + \theta_P^* \theta_Q \right) \rho_P \sigma_P \sigma_Q + \theta_Q \theta_Q^* \sigma_Q^2 \right) dt \\
+ \left( \omega^3 - \omega^2 \right) \left( \theta_P^2 \sigma_P^2 + 2\rho_P \sigma_P \sigma_Q \theta_P \theta_Q + \theta_Q^2 \sigma_Q^2 \right) dt \\
= \omega (1 - \omega) \left( (\theta_P - \theta_P^*) (\mu_R - r) + (\theta_Q - \theta_Q^*) (\mu_Q - r) + c^* - c \right) dt \\
+ \omega (1 - \omega) \left[ (\theta_P - \theta_P^*) \sigma_P dZ_P + (\theta_Q - \theta_Q^*) \sigma_Q dZ_Q \right] \\
+ \omega (1 - \omega)^2 \left( \theta_P^2 \sigma_P^2 + 2\rho_P \sigma_P \sigma_Q \theta_P^* \theta_Q^* + \theta_Q^2 \sigma_Q^2 \right) dt \\
+ \omega (1 - \omega) (2\omega - 1) \left( \theta_P \theta_P^* \sigma_P^2 + \left( \theta_P \theta_Q^* + \theta_P^* \theta_Q \right) \rho_P \sigma_P \sigma_Q + \theta_Q \theta_Q^* \sigma_Q^2 \right) dt \\
- \omega^2 (1 - \omega) \left( \theta_P^2 \sigma_P^2 + 2\rho_P \sigma_P \sigma_Q \theta_P \theta_Q + \theta_Q^2 \sigma_Q^2 \right) dt \\
= \omega (1 - \omega) \left( (\theta_P - \theta_P^*) (\mu_R - r) + (\theta_Q - \theta_Q^*) (\mu_Q - r) + c^* - c \right) dt \\
+ \omega (1 - \omega) \left[ (\theta_P - \theta_P^*) \sigma_P dZ_P + (\theta_Q - \theta_Q^*) \sigma_Q dZ_Q \right] \\
+ \omega (1 - \omega) \left[ (1 - \omega) \theta_P^2 + (2\omega - 1) \theta_P \theta_P^* \omega \theta_P^* \sigma_P^2 dt \right]
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\[ + \omega (1 - \omega) \left[ 2 (1 - \omega) \theta_p^* \theta_Q^* + (2 \omega - 1) \left( \theta_p \theta_Q^* + \theta_p^* \theta_Q \right) - 2 \omega \theta_p \theta_Q \right] \rho_{PQ} \sigma_P \sigma_Q dt \]

\[ + \omega (1 - \omega) \left[ (1 - \omega) \theta_Q^* + (2 \omega - 1) \theta_Q \theta_Q^* - \omega \theta_Q^2 \right] \sigma_Q^2 dt \]

Note that

\[ (1 - \omega) \theta_p^2 + (2 \omega - 1) \theta_p \theta_p^* - \omega \theta_p^2 = - (\theta_p - \theta_p^*) (\omega \theta_p + (1 - \omega) \theta_p^*) = - (\theta_p - \theta_p^*). \]

Similarly,

\[ (1 - \omega) \theta_Q^2 + (2 \omega - 1) \theta_Q \theta_Q^* - \omega \theta_Q^2 = - \left( \theta_Q - \theta_Q^* \right) \left( \omega \theta_Q + (1 - \omega) \theta_Q^* \right) = 0. \]

We can simplify in two ways the expression

\[ 2 (1 - \omega) \theta_p^* \theta_Q^* + (2 \omega - 1) \left( \theta_p \theta_Q^* + \theta_p^* \theta_Q \right) - 2 \omega \theta_p \theta_Q \]

The first way is:

\[ 2 (1 - \omega) \theta_p^* \theta_Q^* + (2 \omega - 1) \left( \theta_p \theta_Q^* + \theta_p^* \theta_Q \right) - 2 \omega \theta_p \theta_Q \]

\[ = 2 (1 - \omega) \theta_p^* \theta_Q^* + (2 \omega - 1) \theta_p \theta_Q^* + (2 \omega - 1) \theta_p^* \theta_Q - 2 \omega \theta_p \theta_Q \]

\[ = \theta_Q^* \left[ 2 (1 - \omega) \theta_p^* + (2 \omega - 1) \theta_p \right] + \theta_Q \left[ (2 \omega - 1) \theta_p^* - 2 \omega \theta_p \right] \]

\[ = (2 - \theta_p) \theta_Q^* + (\theta_p^* - 2) \theta_Q \]

\[ = 2 \left( \theta_Q^* - \theta_Q \right) + \theta_p^* \theta_Q - \theta_p \theta_Q^*; \]

the second way is:

\[ 2 (1 - \omega) \theta_p^* \theta_Q^* + (2 \omega - 1) \left( \theta_p \theta_Q^* + \theta_p^* \theta_Q \right) - 2 \omega \theta_p \theta_Q \]

\[ = 2 (1 - \omega) \theta_p^* \theta_Q^* + (2 \omega - 1) \theta_p \theta_Q^* + (2 \omega - 1) \theta_p^* \theta_Q - 2 \omega \theta_p \theta_Q \]

\[ = \theta_p^* \left[ 2 (1 - \omega) \theta_Q^* + (2 \omega - 1) \theta_Q \right] + \theta_p \left[ (2 \omega - 1) \theta_Q^* - 2 \omega \theta_Q \right] \]

\[ = \theta_p \theta_Q^* - \theta_p \theta_Q. \]

Therefore,

\[ 2 (1 - \omega) \theta_p^* \theta_Q^* + (2 \omega - 1) \left( \theta_p \theta_Q^* + \theta_p^* \theta_Q \right) - 2 \omega \theta_p \theta_Q \]
\[
\begin{align*}
&= \frac{1}{2} \left[ 2 (\theta_Q^* - \theta_Q) + \theta_P \theta_Q - \theta_P \theta_Q^* + \theta_P \theta_Q^* - \theta_P \theta_Q \right] \\
&= \theta_Q^* - \theta_Q.
\end{align*}
\]

This allows us to write the stochastic differential equation for the state variable \( \omega \) as

\[
d\omega = \mu_\omega dt + \sigma_\omega dZ_\omega,
\]

where,

\[
\begin{align*}
\mu_\omega &= \omega (1 - \omega) \left[ (\theta_P - \theta_P^*) \left( \mu_R - \sigma_P^2 - r \right) + (\theta_Q - \theta_Q^*) \left( \mu_Q - \rho P_Q \sigma_P \sigma_Q - r \right) \right. \\
&\left.\left. + \left( \theta_P^* - \theta_P \right) \sigma_P \sigma_Z P + \left( \theta_Q^* - \theta_Q \right) \rho P_Q \sigma_Q dZ_Q \right\} \\
\sigma_\omega dZ_\omega &= \omega (1 - \omega) \left[ (\theta_P - \theta_P^*) \sigma_P dZ_P + \left( \theta_Q - \theta_Q^* \right) \sigma_Q dZ_Q \right].
\end{align*}
\]

### B.2.5 Stock price and price-dividend ratio

Market clearing for the consumption good gives

\[
e = e^* W^* + c W.
\]

Since we have only one share of stock in the market its price \( P \) is given by \( P = W + W^* \).

Hence

\[
e = \omega c + (1 - \omega) e^*.
\]

Thus, the stock price is given by

\[
P = p e,
\]

where,

\[
p = \left[ \omega c + (1 - \omega) e^* \right]^{-1}.
\]

Therefore,

\[
\frac{dP}{P} = \frac{de}{e} + \frac{dp}{p} + \frac{dp}{pe}.
\]

We know that

\[
\frac{de}{e} = \mu_e dt + \sigma_e dZ_e,
\]

and

\[
\frac{dp}{p} = \frac{p_e}{p} d\omega + \frac{p_X}{p} dX + \frac{p_{XX}}{2} dX \cdot dX + \frac{p_{DX}}{p} d\omega \cdot dX + \frac{p_{D\omega}}{p} d\omega \cdot d\omega.
\]
Hence,
\[
\frac{dP}{P} = \left[ \mu_e + \sigma_e \left( \frac{p\omega}{p} \rho_e \omega + \frac{pX}{p} \rho_e X \right) + \mu_\omega \frac{p\omega}{p} + \mu_X \frac{pX}{p} \\
+ \frac{1}{2} \sigma_X^2 \frac{pXX}{p} + \rho_\omega X \sigma_\omega \sigma_X \frac{p\omega X}{p} + \frac{1}{2} \sigma_\omega^2 \frac{p\omega}{p} \right] dt \\
+ \sigma_\omega \frac{p\omega}{p} dZ_\omega + \sigma_X \frac{pX}{p} dZ_X + \sigma_e dZ_e.
\]

Now
\[
\frac{dP + edt}{P} = \frac{dP}{P} + p^{-1} dt.
\]

Therefore,
\[
\frac{dP + edt}{P} = \left[ p^{-1} + \mu_e + \sigma_e \left( \frac{p\omega}{p} \rho_e \omega + \frac{pX}{p} \rho_e X \right) + \mu_\omega \frac{p\omega}{p} + \mu_X \frac{pX}{p} \\
+ \frac{1}{2} \sigma_X^2 \frac{pXX}{p} + \rho_\omega X \sigma_\omega \sigma_X \frac{p\omega X}{p} + \frac{1}{2} \sigma_\omega^2 \frac{p\omega}{p} \right] dt \\
+ \sigma_\omega \frac{p\omega}{p} dZ_\omega + \sigma_X \frac{pX}{p} dZ_X + \sigma_e dZ_e.
\]

Hence,
\[
\frac{dP + edt}{P} = \mu_R dt + \sigma_P dZ_P,
\]

where
\[
\mu_R = p^{-1} + \mu_e + \sigma_e \left( \frac{p\omega}{p} \rho_e \omega + \frac{pX}{p} \rho_e X \right) + \mu_\omega \frac{p\omega}{p} + \mu_X \frac{pX}{p} \\
+ \frac{1}{2} \sigma_X^2 \frac{pXX}{p} + \rho_\omega X \sigma_\omega \sigma_X \frac{p\omega X}{p} + \frac{1}{2} \sigma_\omega^2 \frac{p\omega}{p},
\]

and
\[
\sigma_P dZ_P = \sigma_\omega \frac{p\omega}{p} dZ_\omega + \sigma_X \frac{pX}{p} dZ_X + \sigma_e dZ_e.
\]

Hence,
\[
\sigma_P^2 = \left( \sigma_\omega \frac{p\omega}{p} \right)^2 + \left( \sigma_X \frac{pX}{p} \right)^2 + \sigma_e^2 + 2 \left( \sigma_\omega \sigma_X \frac{p\omega X}{p^2} \rho_e \omega X + \sigma_\omega \sigma_P \frac{p\omega}{p} \rho_e \omega + \sigma_X \sigma_e \frac{pX}{p} \rho_e X \right).
\]

As before, it is straightforward (though a bit more tedious) to obtain the zeroth order expansions in $\alpha$ for the optimal policies and prices because this corresponds to the case
where all investors have log utility. This also allows one to get $g_0$. Using these zero order expressions we can find first order expressions for the equilibrium variables and a partial differential equation for $g_1$ by expanding the Bellman equation in $\alpha$. We can solve this partial differential equation to find $g_1$. We can repeat this procedure as follows: From the $n - 1$ order expressions we compute the equilibrium variables to order $n$ and obtain a partial differential equation for $g_n$. If we can solve this partial differential equation we can compute the $n + 1$ order equilibrium variables and find a partial differential equation for $g_{n+1}$. 
References


