The Role of Risk Aversion and Intertemporal Substitution in Dynamic Consumption-Portfolio Choice with Recursive Utility

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Abstract

The objective of this note is to understand the implications for consumption and portfolio choice of the separation of an investor’s risk aversion and elasticity of intertemporal substitution that is made possible by recursive utility, in contrast to expected utility where the two are dictated by the same parameter. In particular, we study whether the optimal dynamic consumption and portfolio decisions depend on risk aversion, elasticity of intertemporal substitution, or both. We find that, in general, the consumption and portfolio decisions depend on both risk aversion and the elasticity of intertemporal substitution. Only in the case where the investment opportunity set is constant, is the optimal portfolio weight independent of the elasticity of intertemporal substitution, though even in this case the consumption decision depends on both risk aversion and elasticity of intertemporal substitution.

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1 Introduction

Recursive utility functions (Kreps and Porteus, 1978; Epstein and Zin, 1989), in contrast to expected utility functions, enable one to separate cleanly an investor’s risk aversion and elasticity of intertemporal substitution. The objective of this note is to understand the implications of this separation for consumption and portfolio choice.

Our work is motivated by recent research in the areas of finance, for instance Weil (1989) and Campbell and Viceira (2002), and macroeconomics, for example Weil (1990) Obstfeld (1994) and Dumas and Uppal (2001), where the preferences of agents are characterized by recursive utility functions. Explicit analytic solutions for the dynamic consumption-portfolio problem are difficult to obtain for the case where the investment opportunity set is stochastic. For instance, Svensson (1989) and Obstfeld (1994) obtain an explicit solution to the Bellman equation assuming the investment opportunity set is constant. Based on the analysis in a setting with a non-stochastic investment opportunity set, Svensson concludes that:

\[ \text{Hence, the optimal portfolio depends on the risk aversion parameter but not on the intertemporal elasticity of substitution.} \]  

(Svensson 1989, p. 315)

Weil (1990) considers an economy with a stochastic interest rate but in order to get closed-form results he assumes that the interest rate is identically and independently distributed (IID) over time; consequently, just as in the case of a constant investment opportunity set, the optimal portfolio is myopic and does not include a component to hedge against intertemporal changes in the investment opportunity set. Campbell and Viceira (2002) and Chacko and Viceira (1999) allow for non-IID stochastic investment opportunity sets but get analytic expressions that are only approximations to the true consumption and portfolio policies, while Dumas and Uppal (2001) solve their model numerically. Finally, Schroder and Skiadas (1999) characterize the solution to
the continuous time consumption-portfolio problem in terms of a backward stochastic differential equation but do not provide an explicit solution for the case where the investment opportunity set is stochastic. Giovannini and Weil (1989) examine the implications of recursive utility for equilibrium in the capital asset pricing model while Ma (1993) shows the existence of equilibrium in an economy with multiple agents who have recursive utility; but neither paper provides an explicit expression for the optimal consumption and portfolio choice policies with a non-IID stochastic investment opportunity set.

Thus, it is difficult to deduce the properties of the optimal dynamic consumption and portfolio policies from these papers. In particular, these papers do not address the questions raised in this note: (i) does the optimal consumption decision depend on risk aversion, elasticity of intertemporal substitution, or both; (ii) does the optimal portfolio choice depend on risk aversion, elasticity of intertemporal substitution, or both. We consider a simplified discrete-time setting where one can obtain an explicit solution even when the investment opportunity set is stochastic; this setting is similar to the one considered by Ingersoll (1987), but instead of using expected utility we use recursive utility.

Our main result is to show that, in general, the consumption and portfolio decisions depend on both risk aversion and the elasticity of intertemporal substitution. Thus, the result in Svensson (1989) does not generalize to the case where the investment opportunity set is stochastic: only in the special case where the investment opportunity set is constant (or identically and independently distributed over time), the optimal portfolio weight is independent of the elasticity of intertemporal substitution, though even in this case the consumption decision depends on both risk aversion and elasticity of intertemporal substitution. However, the sign of the intertemporal hedging component in the optimal portfolio depends only on the size of risk aversion relative to unity. We also find that the effect of a change in the investment opportunity set on consumption depends on the income and substitution effects, and we show that the relative magnitude of these effects depends on the size of the elasticity of intertemporal substitution relative to unity.

The rest of the paper is organized as follows. The model with recursive utility functions is described in Section 2. The consumption and portfolio problem of the investor is described in the first part of Section 3, with the rest of this section containing the solution to the problem when
the investment opportunity set is constant and when it is stochastic. We conclude in Section 4. Detailed proofs for all propositions are presented in the appendix.

2 The model

In this section, we describe the formulation of recursive utility function that we consider in our analysis. Following this, we describe the financial assets available to the investor, and the dynamic budget constraint that the investor faces when making her consumption and portfolio decisions.

2.1 Preferences

We assume that the agent’s preferences are recursive and of the form described by Epstein and Zin (1989). Hence the agent’s utility at time \( t \), \( U_t \) is given by

\[
U_t = f(c_t, \mu_t(U_{t+1}))
\]

\( U_T = B(W_T, T) \),

where \( f \) is an aggregator function and \( \mu_t(U_{t+1}) \) is a certainty equivalent of the distribution of time \( t + 1 \) utility, \( U_{t+1} \), conditional upon time-\( t \) information. \( B(W_T, T) \) is the bequest function.

We choose the aggregator

\[
f(c, v) = \left[ (1 - \beta) c^\rho + \beta v^\rho \right]^{1/\rho}, \quad \rho \leq 1; \rho \neq 0, \beta > 0
\]

and certainty equivalent

\[
\mu_t[x] = (E_t x^\alpha)^{1/\alpha}, \quad \alpha \leq 1; \alpha \neq 0,
\]

where \( x \) is a random variable. We also choose

\[
B(W_T, T) = (1 - \beta)^{\frac{1}{\rho}} W_T.
\]

Therefore from (2),

\[
U_t = \left[ (1 - \beta) c_t^\rho + \beta (E_t U_{t+1}^\alpha)^{\rho/\alpha} \right]^{1/\rho}.
\]

Note that the relative risk aversion of the agent is given by \( 1 - \alpha \) and the elasticity of intertemporal substitution by \( 1/(1 - \rho) \). Hence, the recursive formulation allows one to disentangle the
effects of relative risk aversion and the elasticity of intertemporal substitution. On the other hand, when \( \rho = \alpha \), equation (5) reduces to the power example of expected utility

\[
U_t = \left[ (1 - \beta) E_t \left( \sum_{j=0}^{T-1} \beta^j C_t^{\alpha} + \beta^{T-t} W_T^{\alpha} \right) \right]^{\frac{1}{\alpha}},
\]

(6)

where relative risk aversion is \( 1 - \alpha \) and the elasticity of intertemporal substitution is \( 1/(1 - \alpha) \), so that both are determined by the same parameter, \( \alpha \).

2.2 Financial Assets

Let there be a riskless asset with return \( R \), and denote the time-\( t \) price of the riskless security by \( P_0(t) \). In addition to the riskless asset, there are \( n \) risky assets, whose rates of return are given by

\[
\frac{P_i(t+1)}{P_i(t)} = z_i(t), \ i \in \{1, \ldots, n\},
\]

(7)

where \( z_i(t) \) is a random variable and \( P_i \) is the price of the \( i \)'th risky asset. Note that the \( t+1 \) price is cum dividend.

2.3 Evolution of wealth and the budget constraint

An investor with wealth \( W(t) \) chooses to consume \( C(t) \) and buy \( N_i(t) \) shares of each asset, subject to the constraint

\[
W(t) - C(t) = I(t) = \sum_{i=0}^{n} N_i(t) P_i(t)
\]

(8)

Hence the proportion of wealth invested in the \( i \)'th asset is given by \( w_i(t) = N_i(t) P_i(t) / I(t) \). Then

\[
W(t+1) = I(t) \sum_{i=0}^{n} w_i(t) z_i(t)
\]

\[
= [W(t) - C(t)] \left[ w_0(t) R + \sum_{i=1}^{n} w_i(t) z_i(t) \right]
\]

\[
= [W(t) - C(t)] \left[ (1 - \sum_{i=1}^{n} w_i(t)) R + \sum_{i=1}^{n} w_i(t) z_i(t) \right]
\]
\[ W(t) - C(t) \left[ \sum_{i=1}^{n} w_i(t) (z_i(t) - R) + R \right] , \quad (9) \]

where the constraint \( \sum_{i=0}^{n} w_i(t) = 1 \) has been substituted out.

3 The intertemporal consumption and portfolio choice problem

The objective of the agent is to maximize her lifetime expected utility by choosing consumption and the proportions of her wealth, \( w_i(t) \), to invest across risky assets, subject to the budget constraint given in equation (9).

3.1 Characterization of the solution to the general problem

We define the optimal value of utility in (5) as a function \( J \) of current wealth, \( W_t \) and time, \( t \). The Bellman equation then takes the form

\[ J(W_t, t) = \sup_{C_t, W_t} \left[ (1 - \beta) C_t^{\rho} + \beta \left( E_t J_{t+1}^{\rho} (W_{t+1}, t + 1) \right)^{\rho/\alpha} \right]^{1/\rho}, \quad (10) \]

where \( w_t = \{w_0(t), ..., w_n(t)\} \) is the vector of portfolio weights.

**Proposition 1** The value function is given by

\[ J(W_t, t) = (1 - \beta)^{\frac{1}{\rho}} a_t^{\frac{\rho-1}{\rho}} W_t \quad (11) \]

where

\[ a_t = \left\{ 1 + \left[ \beta \left( E_t (Z_t)^{\alpha} a_{t+1}^{\frac{\rho}{\rho-1}} \right)^{\frac{\rho}{\rho-1}} \right]^{\frac{1}{1-\rho}} \right\}^{-1}, a_T = 1, \quad (12) \]

and

\[ Z_t = \sum_{i=0}^{n} w_i(t) z_i(t) . \quad (13) \]

The optimal consumption policy is given by

\[ C_t = a_t W_t, \quad (14) \]
and the optimal portfolio policy is defined by the condition
\[ E_t \left[ a_{t+1}^{\alpha (\rho-1)} Z_t^{\alpha-1} (z_i (t) - R) \right] = 0, \] (15)

### 3.2 Solution for a constant investment opportunity set

We first consider the case where the investment opportunity set is constant. This allows us to explicitly identify the value function and the optimal portfolio and consumption choices. We assume that there is only one risky asset with two equally likely payoffs, \( h > R \) and \( k < R \). In this setup, we have the following.

**Proposition 2** The value function is given by
\[ J(W_t, t) = (1 - \beta)^{\frac{1}{\alpha}} a_t^{\frac{\rho-1}{\rho}} W_t, t \in \{1, \ldots, T\}, \] (16)

where
\[ a_t = \frac{1 - \beta v^{\frac{\rho}{\alpha}}}{1 - \left( \beta v^{\frac{\rho}{\alpha}} \right)^{(T-t+1)/(1-\rho)}} \] (17)
\[ v = \frac{1}{2} R^\alpha (h - k)^\alpha \left[ (h - R)^{1-b} + (R - k)^{1-b} \right]^{1-\alpha}. \] (18)

The optimal consumption policy is given by
\[ C_t = a_t W_t, \] (19)

and the optimal portfolio policy by
\[ w_1 (t) = R \frac{(R - k)^{-b} - (h - R)^{-b}}{(R - k)^{1-b} + (h - R)^{1-b}}, \] (20)

where \( b = 1/(1 - \alpha) \).

From equation (19), we see that the investor consumes a fixed proportion of her wealth, which from the definition of \( a_t \) in equation (17), depends on her own rate of time preference, relative risk aversion and intertemporal elasticity of substitution, together with the investment horizon and
characteristics of the risky and riskless assets. Thus, even in the case of constant investment opportunity set, the consumption policy depends on both risk aversion and the elasticity of intertemporal substitution.

On the other hand, the optimal portfolio choice is independent of the investor’s intertemporal elasticity of substitution; the proportion of wealth the investor chooses to invest in the risky asset depends only on the characteristics of the risky and riskless assets and her relative risk aversion.\(^1\) This confirms the result in Svensson (1989).

### 3.3 Stochastic investment opportunity set

We now examine the effects of a stochastic investment opportunity set on optimal portfolio and consumption decisions. We do this by extending to the case of recursive utility an example considered in Ingersoll (1987) for the case of expected utility. This is a simple three-date model with consumption and portfolio decisions at \( t = \{0, 1\} \) and bequest at \( t = 2 \). Moreover, it is assumed that at \( t = 1 \) only a riskless asset is available while at \( t = 0 \) there is also a risky asset that the investor can hold.

In order to understand the results that follow, we start by considering the effect of a change in the investment opportunity set, through a change in the riskless interest rate, on consumption. This effect can be decomposed into “income” and “substitution” effects.

**Proposition 3** The substitution effect is given by

\[
\frac{\partial C_1}{\partial R} \bigg|_{J} = -\frac{a_1^2 W_1 \beta^\frac{1}{1-r} R^{\frac{r}{1-r}} - 1}{1 - \rho} \tag{21}
\]

where

\[
a_1 = \left[ 1 + (\beta R^\rho)^\frac{1}{1-r} \right]^{-1}, \tag{22}
\]

and the income effect by

\[
(W_1 - C_1) R^{-1} \frac{\partial C_1}{\partial W_1} = W_1 (1 - a_1) R^{-1} a_1. \tag{23}
\]

\(^1\)Note that in Proposition 2, by setting \( \rho = \alpha \), we obtain the well known results for power utility and by taking appropriate limits, we obtain the results for logarithmic utility.
The sum of these two effects is given by

$$\frac{dC}{dR} = -a_1^2 W R^{-1} (\beta R^\rho)^{-1} \frac{1}{1-\rho} \rho.$$  \hfill (24)

Note that both the substitution and income effects of a change in the riskfree rate, \( R \), on consumption are independent of the relative risk aversion of the investor. The substitution effect is always negative, whereas the income effect is always positive. This is intuitively clear: as the riskfree rate increases, future consumption becomes cheaper relative to current consumption; hence current consumption decreases (substitution effect). However, the increase in the riskfree rate increases overall wealth, which leads to an increase in current consumption (income effect). When \( \rho > 0 \) (elasticity of intertemporal substitution is greater than unity), the substitution effect is dominant and a rise in the riskfree rate leads to a decrease in current consumption. For the case where \( \rho < 0 \) (elasticity of intertemporal substitution is less than unity), the income effect is dominant and as the riskfree rate rises, current consumption rises. When \( \rho = 0 \), the income effect exactly offsets the substitution effect, which is a well-known result for the case of logarithmic preferences (\( \rho = \alpha = 0 \)). These results are similar to those in Weil (1990).

We now restrict the model further in order to investigate how future changes in the investment opportunity set affect demand for the risky asset. We deliberately define the model such that demand for the risky asset would be zero if the investment opportunity set were constant. Hence, the only demand for the risky asset arises from the desire for intertemporal hedging.

Suppose that at \( t = 0 \) the interest rate which will hold at \( t = 1 \) is unknown. We assume there are two equally probable realizations \( R_D \) and \( R_U \) with \( R_D < R_U \) corresponding to two states \( D \) and \( U \), respectively. At \( t = 0 \) the riskless asset offers a return \( R \) and at time \( t = 1 \) the risky asset gives a return of \( 2R \) if the riskfree rate is \( R_D \) and a return 0 if the riskfree rate is \( R_U \). The expected return on the risky asset is the same as the riskless asset. Therefore, no single-period investor would invest in any of the risky asset. However, multiperiod investors may hold the risky asset in order to hedge against changes in the investment opportunity set.
Proposition 4 The intertemporal hedging demand for the risky asset at time 0 is given by

\[
\begin{align*}
    w_1(0) &= \left[ 1 + (\beta R^D_t)^{1-\rho} \right]^{\frac{\alpha + 1}{1 - \alpha}} - \left[ 1 + (\beta R^U_t)^{1-\rho} \right]^{\frac{\alpha + 1}{1 - \alpha}} \\
    &= \left[ 1 + (\beta R^D_t)^{1-\rho} \right]^{\frac{\alpha + 1}{1 - \alpha}} + \left[ 1 + (\beta R^U_t)^{1-\rho} \right]^{\frac{\alpha + 1}{1 - \alpha}} \tag{25}
\end{align*}
\]

Observe that the optimal portfolio weight depends on both the parameter controlling relative risk aversion, \( \alpha \), and the parameter driving elasticity of intertemporal substitution, \( \rho \). The intuition for why the portfolio weight depends on not just the risk aversion parameter but also on the intertemporal substitution parameter is the following. Because the investment opportunity set is stochastic, the consumption-wealth ratio at time \( t + 1 \), \( a_{t+1} \), and hence consumption, is also stochastic; moreover, the variation in consumption depends on the elasticity of intertemporal substitution. From Cox and Huang (1989) we know that the role of the optimal portfolio is to finance the optimal consumption desired by the investor. Given that the variation in consumption depends on intertemporal substitution, so must the optimal portfolio. On the other hand, when the investment opportunity set is non-stochastic, the consumption-wealth ratio, \( a_{t+1} \), and therefore consumption, are constant and so in this case the optimal portfolio also does not contain a hedging term that would depend on the parameter for intertemporal substitution.

Proposition 5 The hedging demand for the risky asset at \( t = 0 \) is strictly negative if and only if relative risk aversion is strictly less than unity and strictly positive if and only if relative risk aversion is strictly greater than unity. Hedging demand is zero if and only if relative risk aversion is unity.

Thus, the above proposition shows that the sign of hedging demand for the risky asset depends only on the investor’s relative risk aversion. When relative risk aversion is unity (\( \alpha = 0 \)), then the hedging demand for the risky asset is zero. This is a generalization of the result that logarithmic investors, for whom both relative risk aversion and the elasticity of intertemporal substitution are unity, have zero hedging demand.

To understand why the sign of the hedging demand depends on the investor’s relative risk aversion, note that the first order condition for optimal portfolio choice is

\[
E_t \left[ \frac{\partial J^\alpha_{t+1}}{\partial W_{t+1}} (W_{t+1}, t + 1) \right] (z_t (t) - R) = 0. \tag{26}
\]
Equation (26) shows that the optimal portfolio maximizes the conditional expected marginal utility weighted by the excess return on the risky asset. The expression for the value function in equation (16) allows one to rewrite equation (26) as done in equation (15), which we reproduce below:

$$E_t \left[ \alpha \left( \frac{\rho - 1}{\rho} \right) Z_t^{\alpha - 1} (z_i (t) - R) \right] = 0.$$  

Writing the above expectation explicitly after substituting in the returns on the assets held in the portfolio, we obtain:

$$\left[ a_1 (D) \right]^{\alpha (p-1) \rho} \left[ 1 + w_1 (0) \right]^{\alpha - 1} = \left[ a_1 (U) \right]^{\alpha (p-1) \rho} \left[ 1 - w_1 (0) \right]^{\alpha - 1}$$

(27)

Note that $\left[ a_1 (D) \right]^{\alpha (p-1) \rho} < \left[ a_1 (U) \right]^{\alpha (p-1) \rho}$ if and only if $RRA < 1$, while $\left[ a_1 (D) \right]^{\alpha (p-1) \rho} > \left[ a_1 (U) \right]^{\alpha (p-1) \rho}$ if and only if $RRA > 1$, and the two terms are equal if and only if $RRA = 1$. Hence, for the case where relative risk aversion is strictly less than unity, if $w_1 (0) = 0$ then the marginal utility of wealth in the up ($U$) state is lower than in the down ($D$) state. To hedge against this, the agent will short the stock in order to increase marginal utility in the up state. On the other hand, for the case where relative risk aversion is strictly greater than unity, if $w_1 (0) = 0$ then the marginal utility of wealth in the down state is lower than in the up state, and to hedge against this the agent will go long the stock so that the marginal utility in the down state is increased.

4 Conclusion

We study the role of risk aversion and intertemporal substitution in the optimal consumption and portfolio problem of an individual investor when the investment opportunity set is stochastic. Using a simple example that admits a solution in closed form we show that the consumption and portfolio decisions depend on both risk aversion and the elasticity of intertemporal substitution. Only for the special case where the investment opportunity set is constant is the optimal portfolio weight independent of the elasticity of intertemporal substitution, though even in this case the consumption decision depends on both risk aversion and elasticity of intertemporal substitution.
Proofs

Proof of Proposition 1

We have the Bellman equation

\[ J(W_t, t) = \sup_{C_t, W_t} \left[ (1 - \beta) C_t^\rho + \beta \left( E_t J_{t+1}^\alpha (W_{t+1}, t+1) \right)^{\rho/\alpha} \right]^{1/\rho} \]

and the intertemporal budget constraint

\[ W(t+1) = I(t) \left[ \sum_{i=1}^n w_i(t) (z_i(t) - R) + R \right] = I(t) Z(t), \]

where \( Z(t) = \sum_{i=1}^n w_i(t) (z_i(t) - R) + R. \) Hence we can derive the first order conditions for optimality. The first order condition for consumption is given by

\[ \frac{\partial}{\partial C_t} \left[ (1 - \beta) C_t^\rho + \beta \left( E_t J_{t+1}^\alpha (I_t Z_t, t+1) \right)^{\rho/\alpha} \right]^{1/\rho} = 0. \]

Hence optimal consumption is given by

\[ C_t = \left\{ \frac{\beta}{1 - \beta} \left( E_t J_{t+1}^\alpha (I_t Z_t, t+1) \right)^{\rho/\alpha - 1} E_t \left[ J_{t+1}^{\alpha - 1} (W_{t+1}, t+1) \frac{\partial J_{t+1} (W_{t+1}, t+1)}{\partial W_{t+1}} Z_t \right] \right\}^{\frac{1}{\rho - 1}}. \quad (28) \]

The first order condition for the optimal portfolio is given by

\[ \frac{\partial}{\partial w_{it}} \left[ (1 - \beta) C_t^\rho + \beta \left( E_t J_{t+1}^\alpha (I_t Z_t, t+1) \right)^{\rho/\alpha} \right]^{1/\rho} = 0. \]

This can be rewritten as

\[ E_t \left[ J_{t+1}^{\alpha - 1} (W_{t+1}, t+1) \frac{\partial J_{t+1} (W_{t+1}, t+1)}{\partial W_{t+1}} (z_i(t) - R) \right] = 0. \quad (29) \]

Now we consider the expression

\[ J(W_t, t) = \left[ (1 - \beta) C_t^\rho + \beta \left( E_t J_{t+1}^\alpha (W_{t+1}, t+1) \right)^{\rho/\alpha} \right]^{1/\rho}, \]

where \( W_{t+1} = I_t Z_t \).
and

\[ I_t = W_t - C_t, \]
\[ Z_t = \sum_{i=1}^{n} w_i(t) (z_i(t) - R) + R. \]

We calculate the derivative

\[
\frac{\partial J(W_t, t)}{\partial W_t} = \frac{1}{\rho} \left[ (1 - \beta) C_t^\rho + \beta (E_t J^\alpha (W_{t+1}, t+1))^{\rho/\alpha} \right]^{1/\rho - 1} \frac{\partial}{\partial W_t} [(1 - \beta) C_t^\rho + \beta (E_t J^\alpha (W_{t+1}, t+1))^{\rho/\alpha}].
\]

The above expression can be simplified to obtain

\[
\frac{\partial J(W_t, t)}{\partial W_t} = J^{1-\rho} (W_t, t) [(1 - \beta) C_t^{\rho-1} \frac{\partial C_t}{\partial W_t} + \beta (E_t J^\alpha (W_{t+1}, t+1))^{\rho/\alpha-1} E_t J^{\alpha-1}(W_{t+1}, t+1) \frac{\partial J(W_{t+1}, t+1)}{\partial W_{t+1}} \frac{\partial W_{t+1}}{\partial W_t}].
\]

Now

\[
\frac{\partial W_{t+1}}{\partial W_t} = \frac{\partial (I_t Z_t)}{\partial W_t} = I_t \frac{\partial Z_t}{\partial W_t} + Z_t \left( \frac{\partial I_t}{\partial W_t} + \frac{\partial I_t}{\partial C_t} \frac{\partial C_t}{\partial W_t} \right)
\]
\[
= I_t \sum_{i=1}^{n} \frac{\partial Z_t}{\partial w_i} \frac{\partial w_i}{\partial W_t} + Z_t \left( \frac{\partial I_t}{\partial W_t} + \frac{\partial I_t}{\partial C_t} \frac{\partial C_t}{\partial W_t} \right)
\]
\[
= I_t \sum_{i=1}^{n} (z_i(t) - R) \frac{\partial w_i}{\partial W_t} + Z_t (1 - \frac{\partial C_t}{\partial W_t})
\]

Therefore, using the first order conditions we can show that

\[
\frac{\partial J(W_t, t)}{\partial W_t} = \beta J^{1-\rho} (W_t, t) [E_t J^\alpha (W_{t+1}, t+1)]^{\rho/\alpha-1} E_t \left[ J^{\alpha-1}(W_{t+1}, t+1) \frac{\partial J(W_{t+1}, t+1)}{\partial W_{t+1}} \frac{\partial W_{t+1}}{\partial W_t} \right].
\]

Recall that optimal consumption is given by equation (28). Hence,

\[
C_t = \left[ \frac{1}{1 - \beta} \frac{\partial J(W_t, t)}{\partial W_t} \right]^{\frac{1}{\rho-1}} = \left[ \frac{1}{1 - \beta} \frac{\partial J(W_t, t)}{\partial W_t} \right]^{\frac{1}{\rho-1}} J(W_t, t).
\]

(30)

Now we can simplify the expression

\[
J(W_t, t) = \sup_{C_t, W_t} \left[ (1 - \beta) C_t^\rho + \beta (E_t J^\alpha (W_{t+1}, t+1))^{\rho/\alpha} \right]^{1/\rho}
\]
to obtain an equation for the value function,

\[ J(W_t, t) = \left(1 - \beta \right) \left[ 1 - \beta \frac{\partial J(W_t, t)}{\partial W_t} \right]^{1/\rho} J^\rho(W_t, t) + \beta [E_t J^\alpha(W_{t+1}, t+1)]^{\rho/\alpha} \]. \tag{31} \]

We seek a trial solution of the form

\[ J(W_t, t) = Ah_t W_t, \]

where

\[ A = (1 - \beta)^{\frac{1}{\rho}} \]
\[ h_T = 1 \]

to ensure that

\[ J(W_T, T) = (1 - \beta)^{\frac{1}{\rho}} W_T \equiv B(W_T, T). \]

Therefore,

\[ \frac{\partial J(W_t, t)}{\partial W_t} = Ah_t. \]

Hence, from equation (30)

\[ C_t = (1 - \beta)^{\frac{1}{1 - \rho}} (Ah_t)^{\frac{1}{1 - \rho}} W_t \]

We now simplify the last term in (31)

\[ \beta [E_t J^\alpha(W_{t+1}, t+1)]^{\rho/\alpha} = \beta [A^\alpha h_t^\alpha W_t^\alpha]^{\rho/\alpha} \]

\[ W_{t+1}^\alpha = (W_t - C_t)^\alpha Z_t^\alpha \]
\[ = \left[ 1 - (1 - \beta)^{\frac{1}{1 - \rho}} (Ah_t)^{\frac{1}{1 - \rho}} \right] W_t^\alpha Z_t^\alpha \]

Define

\[ a_t = (1 - \beta)^{\frac{1}{1 - \rho}} (Ah_t)^{\frac{1}{1 - \rho}} \]
\[ = h_t^{\frac{1}{1 - \rho}} \]

Then

\[ (1 - \beta) C_t^\rho = (1 - \beta) a_t^\rho W_t^\rho \]

and

\[ Ah_t = a_t^{\frac{\rho - 1}{\rho}} (1 - \beta)^{\frac{1}{\rho}}. \]
Thus, we can write the value function as 

\[ J(W_t, t) = (1 - \beta)\frac{1}{\rho} a_t \frac{\pi}{\rho - 1} W_t. \]

Hence

\[
\beta \left( E_t J^\alpha_{t+1} (W_{t+1}, t + 1) \right)^{\rho/\alpha} = \beta \left[ E_t (Ah_{t+1})^{\alpha} W_{t+1}^\alpha \right]^{\frac{\rho}{\alpha - 1}}
\]

\[ = \beta \left[ E_t a_{t+1}^\frac{\rho}{\alpha - 1} (1 - \beta)^{\frac{1}{\alpha - 1}} W_{t+1}^\alpha \right]^{\frac{\rho}{\alpha - 1}}
\]

\[ = \beta (1 - \beta) \left[ E_t a_{t+1}^\frac{\rho}{\alpha - 1} W_{t+1}^\alpha \right]^{\frac{\rho}{\alpha - 1}}. \]

Note that

\[ W_{t+1}^\alpha = \left[ 1 - (1 - \beta) \frac{1}{\rho - 1} (Ah_t)^{\frac{\rho}{\alpha - 1}} \right]^{\alpha} W_t^\rho Z_t^\alpha. \]

Therefore

\[
\beta \left[ E_t J^\alpha_{t+1} (W_{t+1}, t + 1) \right]^{\rho/\alpha} = \beta (1 - \beta) (1 - a_t) \left[ E_t a_{t+1}^\frac{\rho}{\alpha - 1} Z_t^\alpha \right]^{\frac{\rho}{\alpha - 1}} W_t^\rho.
\]

Then, equation (31) implies that

\[
(1 - \beta) \frac{1}{\rho - 1} a_t \frac{\pi}{\rho - 1} W_t = \left[ (1 - \beta) a_t^\rho W_t^\rho + \beta (1 - \beta) (1 - a_t) \left[ E_t a_{t+1}^\frac{\rho}{\alpha - 1} Z_t^\alpha \right]^{\frac{\rho}{\alpha - 1}} W_t^\rho \right]^{\frac{1}{\rho - 1}}.
\]

We can simplify this expression to obtain

\[ a_t = \left\{ 1 + \left[ \beta \left( E_t (Z_t^\alpha a_{t+1}^{\frac{\rho}{\alpha - 1}}) \right)^{\frac{\rho}{\alpha - 1}} \right]^{\frac{1}{\rho - 1}} \right\}^{-1}. \]

We know that the first order condition for the optimal portfolio condition is

\[
E_t \left[ J_t^{\alpha_{t+1} - 1} (W_{t+1}, t + 1) \frac{\partial J_{t+1}}{\partial W_{t+1}} (z_i(t) - R) \right] = 0. \tag{32}
\]

This can now be simplified to obtain

\[
E_t \left[ Z_t^{\alpha_{t+1} - 1} h_{t+1}^\alpha (z_i(t) - R) \right] = 0.
\]

Therefore the first order condition for the optimal portfolio is

\[
E_t \left[ Z_t^{\alpha_{t+1} - 1} a_{t+1}^{\frac{\pi}{\rho - 1}} (z_i(t) - R) \right] = 0.
\]

\[ \blacksquare \]
Proof of Proposition 2

Given that we assume the opportunity set is constant, \( a_{t+1} \) is known at time \( t \). Hence

\[
a_t^{-1} = 1 + \left( \beta v^\frac{1}{1-\rho} \right) a_{t+1}^{-1},
\]

where

\[
v = E_t (Z_t)^\alpha.
\]

Note that because the opportunity set is constant, \( v \) is independent of time. We can solve the difference equation (33) to obtain

\[
a_t = \frac{1 - \beta v^\frac{1}{1-\rho}}{1 - \left( \beta v^\frac{1}{1-\rho} \right)^{(T-t+1)/(1-\rho)}},
\]

where we have used the terminal condition

\[
a_T = 1.
\]

We now only have one risky asset with two equally likely payoffs. Therefore we can simplify the optimal portfolio condition to obtain

\[
E_t \left[ Z_t^{\alpha-1} (z_1 (t) - R) \right] = 0
\]

\[
E_t \left[ (w_0 (t) R + w_1 (t) z_1 (t))^{\alpha-1} (z_1 (t) - R) \right] = 0
\]

\[
E_t \left[ ((1 - w_1 (t)) R + w_1 (t) z_1 (t))^{\alpha-1} (z_1 (t) - R) \right] = 0
\]

\[
E_t \left[ (R + w_1 (t) [z_1 (t) - R])^{\alpha-1} (z_1 (t) - R) \right] = 0
\]

\[
\frac{1}{2} \left[ [R + w_1 (t) (h - R)]^{\alpha-1} (h - R) + [R + w_1 (t) (k - R)]^{\alpha-1} (k - R) \right] = 0
\]

\[
[R + w_1 (t) (h - R)]^{\alpha-1} (h - R) + [R + w_1 (t) (k - R)]^{\alpha-1} (k - R) = 0
\]

Hence, the optimal portfolio is given by

\[
w_1 (t) = R \frac{(R - k)^{-b} - (h - R)^{-b}}{(R - k)^{-b} + (h - R)^{-b}},
\]

where

\[
b = \frac{1}{1 - \alpha}.
\]

We can also simplify the expression for

\[
v = E_t (Z_t)^\alpha,
\]
Hence

\[ v = E_t (w_0 (t) R + w_1 (t) z_1 (t))^\alpha \]
\[ = E_t [R + w_1 (t) (z_1 (t) - R)]^\alpha \]
\[ = \frac{1}{2} \{ [R + w_1 (t) (h - R)]^\alpha + [R + w_1 (t) (k - R)]^\alpha \} \]

Simplifying the first term in the above expression yields

\[ R + w_1 (t) (h - R) = R + R (h - R) \frac{(R - k)^{1-b} - (h - R)^{-b}}{(R - k)^{1-b} + (h - R)^{1-b}} \]
\[ = R \frac{(R - k)^{1-b} + (h - R)^{1-b} + (h - R) (R - k)^{-b} - (h - R)^{1-b}}{(R - k)^{1-b} + (h - R)^{1-b}} \]
\[ = R \frac{(R - k)^{1-b} + (h - R) (R - k)^{-b}}{(R - k)^{1-b} + (h - R)^{1-b}} \]
\[ = R \frac{(R - k)^{-b} (h - k)}{(R - k)^{1-b} + (h - R)^{1-b}}. \]

Similarly,

\[ R + w_1 (t) (k - R) = R + R (k - R) \frac{(R - k)^{-b} - (h - R)^{-b}}{(R - k)^{1-b} + (h - R)^{1-b}} \]
\[ = R \frac{(R - k)^{1-b} + (h - R)^{1-b} - (R - k)^{1-b} - (k - R) (h - R)^{-b}}{(R - k)^{1-b} + (h - R)^{1-b}} \]
\[ = R \frac{(h - R)^{1-b} - (k - R) (h - R)^{-b}}{(R - k)^{1-b} + (h - R)^{1-b}} \]
\[ = R \frac{(h - R)^{-b} (h - k)}{(R - k)^{1-b} + (h - R)^{1-b}}. \]

Hence,

\[ v = \frac{1}{2} R^\alpha (h - k)^\alpha \left[ (h - R)^{1-b} + (R - k)^{1-b} \right]^{1-\alpha} . \]
Proof of Proposition 3

The standard two-good cross-substitution Slutsky equation, which measures the change in the quantity of the \( i \)'th good, \( Q_i \), with respect to a change in the price of the \( j \)'th good, \( P_j \), is

\[
\frac{dQ_i}{dP_j} = \frac{\partial Q_i}{\partial P_j} \bigg|_U - Q_j \frac{\partial Q_i}{\partial W} \bigg|_{R,P_j}.
\]

The first term is the substitution effect and the second is the income effect. At \( t = 1 \), \( Q_1 = C_1 \), \( Q_2 = W_2 \), \( P_1 = 1 \), and \( P_2 = R^{-1} \). Substituting into the above expression and using \( \frac{dP_2}{dR} = -R^{-2} \) we obtain

\[
\frac{dC_1}{dR} = \frac{dP_2}{dR} \frac{dC_1}{dP_2} = \frac{dP_2}{dR} \left[ \frac{\partial C_1}{\partial R} \bigg|_j \frac{\partial R}{\partial P_2} - W_2 \frac{\partial C_1}{\partial W_1} \bigg|_R \right]
\]

\[
= -R^{-2} \left[ - \frac{\partial C_1}{\partial R} \bigg|_j R^2 - W_2 \frac{\partial C_1}{\partial W_1} \bigg|_R \right]
\]

\[
= \frac{\partial C_1}{\partial R} \bigg|_j + W_2 R^{-2} \frac{\partial C_1}{\partial W_1} \bigg|_R
\]

\[
= \frac{\partial C_1}{\partial R} \bigg|_j + (W_1 - C_1) R^{-1} \frac{\partial C_1}{\partial W_1} \bigg|_R.
\]

We know that

\[
J(W_1,1) = (1 - \beta) \frac{1}{\rho} a_1^{\frac{1}{\rho} - 1} W_1 = (1 - \beta) \frac{1}{\alpha} a_1^{\frac{1}{\alpha} - 1} C_1 = (1 - \beta) \frac{1}{\alpha} a_1^{\frac{1}{\alpha}} C_1.
\]

Hence

\[
\frac{\partial J}{\partial C_1} = (1 - \beta) \frac{1}{\alpha} a_1^{-\frac{1}{\alpha}}
\]

and

\[
\frac{\partial J}{\partial R} = -\frac{1}{\rho} (1 - \beta) \frac{1}{\alpha} a_1^{-\frac{1}{\alpha} - 1} \frac{\partial a_1}{\partial R} C_1.
\]

From the Implicit Function Theorem

\[
\frac{\partial C_1}{\partial R} \bigg|_j = -\frac{\partial J/\partial R}{\partial J/\partial C_1}.
\]

We know that

\[
a_t = \left\{ 1 + \left[ \beta \left( E_t (Z_t)^{\alpha} a_{t+1}^{2(\rho-1)} \right)^\frac{1}{\rho} \right] \right\}^{-1}.
\]

Now \( w_1(1) = 0 \). Therefore \( Z_1 = R \). Hence

\[
a_1^{-1} = 1 + \left[ \beta \left( E_1 (R)^{\alpha} a_2^{2(\rho-1)} \right)^\frac{1}{\rho} \right]^{\frac{1}{\rho}}
\]
\begin{align*}
  &= 1 + \left[ \beta \left( E_1 (R)^\alpha \right)^\frac{\beta}{1+\rho} \right]^{\frac{1}{1+\rho}} \text{ because } a_2 = 1 \\
  &= 1 + (\beta R^\rho)^{\frac{1}{1+\rho}} \\

\end{align*}

We calculate

\begin{align*}
\frac{\partial J}{\partial R} / \frac{\partial J}{\partial C_1} &= -\frac{1}{\rho} (1 - \beta)^{\frac{\beta}{1+\rho}} a_1^{\frac{1}{1+\rho}-1} \frac{\partial a_1}{\partial R} C_1 \\
&= -\frac{1}{\rho} C_1 \frac{\partial a_1}{\partial R} \\
&= C_1 \frac{\partial \log a_1^{-1}}{\partial R}.
\end{align*}

From

\begin{align*}
a_1^{-1} = 1 + (\beta R^\rho)^{\frac{1}{1+\rho}},
\end{align*}

we obtain

\begin{align*}
\frac{\partial \log a_1^{-1}}{\partial R} &= \frac{1}{1 + (\beta R^\rho)^{\frac{1}{1+\rho}}} \frac{\partial \left[ (\beta R^\rho)^{\frac{1}{1+\rho}} \right]}{\partial R} \\
&= \frac{\rho}{1 - \rho} \frac{1}{1 + (\beta R^\rho)^{\frac{1}{1+\rho}}} R^{\frac{1}{1+\rho} - 1} \\
&= \frac{\rho \beta^{\frac{1}{1+\rho}} a_1 R^{\frac{1}{1+\rho} - 1}}{1 - \rho}.
\end{align*}

Hence

\begin{align*}
\frac{\partial J}{\partial R} / \frac{\partial J}{\partial C_1} &= \frac{a_1 C_1 \beta^{\frac{1}{1+\rho}} R^{\frac{1}{1+\rho} - 1}}{1 - \rho}.
\end{align*}

Therefore we obtain the substitution effect

\begin{align*}
\frac{\partial C_1}{\partial R} \bigg|_J &= -\frac{a_1^2 W_1 \beta^{\frac{1}{1+\rho}} R^{\frac{1}{1+\rho} - 1}}{1 - \rho},
\end{align*}

where

\begin{align*}
a_1^{-1} = 1 + (\beta R^\rho)^{\frac{1}{1+\rho}}.
\end{align*}

The income effect is

\begin{align*}
(W_1 - C_1) R^{-\alpha} \frac{\partial C_1}{\partial W_1} = W_1 (1 - a_1) R^{-1} a_1.
\end{align*}
Hence, the net effect is
\[
\frac{dC}{dR} = \frac{a_1^2 W_1 \beta^{\frac{1}{1-\rho}} R^{\frac{1}{1-\rho}}}{1-\rho} + W_1 (1 - a_1) R^{-1} a_1
\]
\[
= a_1^2 W R^{-1} \left[ -\frac{\beta^{\frac{1}{1-\rho}} R^{\frac{1}{1-\rho}}}{1-\rho} + (\beta R^\rho)^{\frac{1}{1-\rho}} \right]
\]
\[
= -a_1^2 W R^{-1} (\beta R^\rho)^{\frac{1}{1-\rho}} \left( \frac{1}{1-\rho} - 1 \right)
\]
\[
= -a_1^2 W R^{-1} (\beta R^\rho)^{\frac{1}{1-\rho}} \frac{\rho}{1-\rho}.
\]

Note that at time 1, the investor’s value function is given by
\[
J(W_1, 1) = (1 - \beta)^{\frac{1}{1-\rho}} \left[ 1 + (\beta R^\rho)^{\frac{1}{1-\rho}} \right]^{\frac{1-\rho}{\rho}} W_1.
\]

**Proof of Proposition 4**

The first order condition for the optimal portfolio is
\[
E_t \left[ Z_t^{\alpha-1} a_{t+1}^{\alpha(a-1)\sigma} (z_i (t) - R) \right] = 0
\]

We know that
\[
a_1 = \begin{cases} 
1 + (\beta R_D^\sigma)^{\frac{1}{1-\rho}} & \text{with probability } 1/2 \\
1 + (\beta R_U^\sigma)^{\frac{1}{1-\rho}} & \text{with probability } 1/2 
\end{cases}
\]

Define
\[
a_1(D) = \left[ 1 + (\beta R_D^\rho)^{\frac{1}{1-\rho}} \right]^{-1}
\]
\[
a_1(U) = \left[ 1 + (\beta R_U^\rho)^{\frac{1}{1-\rho}} \right]^{-1}.
\]

We now simplify the expression
\[
E_t \left[ Z_t^{\alpha-1} a_{t+1}^{\alpha(a-1)} (z_i (t) - R) \right] = 0
\]

\[
E_0 \left[ a_1^{\frac{\alpha(a-1)}{\rho}} Z_0^{\alpha-1} (z_1 (0) - R) \right] = 0
\]
\[
\frac{1}{2} \left\{ a_1^{\alpha - 1} (D) [(1 - w_1(0)) R + w_1(0) 2 R^{\alpha - 1} (2R - R) + a_1^{\alpha - 1} (U) [(1 - w_1(0)) R^{\alpha - 1} (-R)] \right\} = 0 \\
\left\{ a_1^{\alpha - 1} (D) [(1 - w_1(0)) R + w_1(0) 2 R^{\alpha - 1} (2R - R) + a_1^{\alpha - 1} (U) [(1 - w_1(0)) R^{\alpha - 1} (-R)] \right\} = 0 \\
a_1^{\alpha - 1} (D) [(1 + w_1(0))]^{\alpha - 1} - a_1^{\alpha - 1} (U) [(1 - w_1(0)) R^{\alpha - 1} = 0 \\
\left[ a_1^{\alpha - 1} (D) + a_1^{\alpha - 1} (U) \right] w_1(0) = a_1^{\alpha - 1} (U) - a_1^{\alpha - 1} (D)
\]

Therefore

\[
w_1(0) = \frac{\left[ 1 + (\beta R^0_D)^{\frac{1}{1+\rho}} \right]^{\alpha - 1} - \left[ 1 + (\beta R^0_U)^{\frac{1}{1+\rho}} \right]^{\alpha - 1}}{\left[ 1 + (\beta R^0_D)^{\frac{1}{1+\rho}} \right]^{\alpha - 1} + \left[ 1 + (\beta R^0_U)^{\frac{1}{1+\rho}} \right]^{\alpha - 1}}.
\]

\[\blacksquare\]

**Proof of Proposition 5**

The hedging demand for the risky asset is given by

\[
w_1(0) = \frac{\left[ 1 + (\beta R^0_D)^{\frac{1}{1+\rho}} \right]^{\alpha - 1} - \left[ 1 + (\beta R^0_U)^{\frac{1}{1+\rho}} \right]^{\alpha - 1}}{\left[ 1 + (\beta R^0_D)^{\frac{1}{1+\rho}} \right]^{\alpha - 1} + \left[ 1 + (\beta R^0_U)^{\frac{1}{1+\rho}} \right]^{\alpha - 1}}
\]

Note that this expression is strictly negative iff

\[
\frac{\rho}{1 - \rho} > 0 \quad \text{and} \quad \frac{\alpha (\rho - 1)}{\rho (\alpha - 1)} > 0 \quad \text{or}
\]

\[
\frac{\rho}{1 - \rho} < 0 \quad \text{and} \quad \frac{\alpha (\rho - 1)}{\rho (\alpha - 1)} < 0
\]

Note further that

\[
\frac{\rho}{1 - \rho} > 0 \quad \text{and} \quad \frac{\alpha (\rho - 1)}{\rho (\alpha - 1)} > 0 \iff
\]

\[
\frac{\rho}{1 - \rho} > 0 \quad \text{and} \quad \frac{\alpha}{1 - \alpha} > 0 \iff
\]

\[
0 < \rho, \alpha < 1
\]
and

\[
\frac{\rho}{1 - \rho} < 0 \text{ and } \frac{\alpha (\rho - 1)}{\rho (\alpha - 1)} < 0 \iff
\]

\[
\frac{\rho}{1 - \rho} < 0 \text{ and } \frac{\alpha}{1 - \alpha} > 0
\]

\[\rho < 0 \text{ or } \rho > 1 \text{ and } 0 < \alpha < 1\]

Hence it is clear the hedging demand for the risky asset is strictly negative iff $0 < \alpha < 1$ and strictly positive iff $\alpha < 0$. The case $\alpha \geq 1$ is excluded by our assumption in the definition of the utility function that $\alpha < 1$. This gives us Proposition 5.
References


