Expected Returns and the Business Cycle: Heterogeneous Agents and Heterogeneous Goods

Lars A. Lochstoer, London Business School*

May 10, 2006

Abstract

I document that expected excess stock market returns contain both low-frequency and higher-frequency components. The low-frequency variation is associated with price-related variables such as the dividend/price ratio, while the higher-frequency variation is associated with business cycle variables such as GDP growth. The two components interact. In particular, when the low-frequency component of the equity premium is high, the impact of business cycle fluctuations on the equity premium is stronger. I propose a simple explanation: Fundamental risks in the economy fluctuate at a business cycle frequency, while the risk tolerance of the marginal investor varies at a lower, "generational" frequency. I construct a general equilibrium model with heterogeneous agents and heterogeneous goods that can account for the new empirical findings. The equilibrium interaction of the dynamic allocation of wealth across agents and of consumption across goods give rise to the empirical relation between the low-frequency price variables, the higher-frequency business-cycle variables, and future excess stock market returns.

*This paper is a revised version of Chapter 1 of my dissertation at the Haas School of Business, University of California at Berkeley. I thank Pierre Collin-Dufresne, Roger Craine, Tom Davidoff, Keener Hughen, Michael Jansson, Georg Kaltenbrunner, Shimon Kogan, Jacob Sagi, Richard Stanton, Motohiro Yogo and especially my adviser Greg Duffee for helpful comments. In addition, I thank seminar participants at Berkeley, Boston College, Carnegie-Mellon, CEPR Gerzensee 2005, Columbia, London Business School, Norwegian School of Economics, Oslo School of Management (BI), Stanford, University of British Columbia, University of Southern California, University of Washington at Seattle, and the WFA 2005. I thank the Dean Witter Foundation and Senter for Pengepolitisk og Finansiell Forskning for financial support. All errors are mine alone. Contact info: Lars Lochstoer, P216, London Business School, Regent’s Park, London, NW1 4SA, U.K. Tel: +44 (0) 7262 5050, ext. 3360, E-mail: LLochstoer@london.edu
1 Introduction

Consumption-based asset-pricing models tie the dynamic behavior of the stock market risk premium to the real economy. In order to fit the unconditional moments of aggregate consumption growth and asset returns, these models generally predict a counter-cyclical risk premium (e.g., Chan and Kogan (2002), Campbell and Cochrane (1999)). On an intuitive level, the cycle that one refers to is the business cycle. However, the equity premium generated by these models is typically extremely slow-moving; it essentially follows a "generational" cycle too long to reflect a pure (NBER) business cycle phenomenon. In fact, empirically there is little evidence so far of a tight link between the state of the economy (the business cycle) and measures of the equity risk premium, although these are usually correlated. In particular, the predictability of excess equity market returns appears to be strong at forecasting horizons much longer than the length of a typical business cycle, and some of the most successful forecasting variables, like the price-dividend ratio or the short-term interest rate, are much more persistent than typical business cycle variables.

Should we then conclude that there is no important relation between the conditional equity premium and business cycle fluctuations? We should not. This paper provides new, strong evidence that traditional business cycle variables such as GDP growth do forecast excess returns. I document significant predictive power after controlling nonlinearly for standard, more persistent predictors of the equity premium (e.g., the dividend yield, or the consumption-wealth ratio). I find that: (i) The business cycle is related to higher-frequency variations in the risk premium, (ii) the relation has the expected sign, that is the risk premium is higher in recessions, and (iii) business cycle fluctuations have a bigger impact on the risk premium when the aggregate risk premium is otherwise high, as predicted by the more persistent, price-related forecasting variables. I conclude that the equity premium has both a business cycle component as well as a generational component and that the two components interact.

I propose an intuitive explanation for this new evidence. First, price-related forecasting variables proxy for the marginal agent’s risk aversion since prices are low when discount

---

1I use "generational" as a term for cycles that are substantially longer than standard business cycles, e.g. 15-20 year cycles, or with a persistence corresponding to the post-war persistence of the aggregate price/dividend ratio.
rates are high. Second, the impact of fluctuations in fundamental risk on asset prices is increasing in this agent’s level of risk aversion. Therefore, if the marginal agent must bear more fundamental risk in recessions than in expansions, business cycle variables will be stronger forecasting variables when, say, the dividend-price ratio is high.

I propose a two-good, heterogeneous agents economy that formalizes this intuition and can account for the empirical findings. In the model, the dynamic allocation of wealth between heterogeneous agents, which determines the relative risk aversion of the marginal agent, operates at a generational frequency. The dynamic relative consumption of different goods, which determines the amount of consumption volatility the marginal agent must bear, operates at a business cycle frequency.

In particular, I assume an exchange economy where agents have non-homothetic preferences over luxury goods and basic goods, as in Ait-Sahalia, Parker and Yogo (2004). Basic good consumption is subject to a common, external minimum consumption level, while luxury good consumption is not. As a result, the wealthier an agent is, the larger the fraction of luxury goods to basic goods he would like to consume. Positive shocks (expansions) therefore increase the aggregate relative level of luxury good consumption in the economy. I show that aggregate luxury good consumption growth in the model and in the data is pro-cyclical with high and counter-cyclical volatility. This counter-cyclical volatility is what generates higher fundamental risk in recessions relative to expansions and is the source of business cycle variation in the equity premium in the model.

I furthermore assume investors to be heterogeneous in their coefficients of relative risk aversion. The more risk averse agents desire lower consumption volatility and therefore follow portfolio strategies that make them relatively wealthy when aggregate consumption is low. On the other hand, when aggregate consumption is high the less risk averse will be the wealthier. Thus, the wealth-weighted average level of risk aversion in the economy varies over time. As in Chan and Kogan (2002), all agents are exposed to a slow-moving "standard of living" process (a ratio habit), which is the benchmark for whether aggregate consumption is high or low. This process is calibrated to a generational frequency, and the time-varying risk aversion of the marginal agent therefore follows a generational cycle. This is the source of the highly persistent component of the equity premium.

In the model, the business cycle component and the highly persistent, generational component of the equity premium interact because of the following reason. If the average investor is very risk averse, changes in the volatility of luxury good consumption growth (the business cycle variable) have a large effect on asset prices. If, on the other hand, the average investor
is close to risk neutral, changing volatility does not matter much. Thus, the components
are multiplicative and the model implies a risk premium that is a nonlinear function of both
the very persistent component and the business cycle component. I confirm the model’s
main time-series predictions using available data on aggregate luxury good consumption,
the business cycle component of aggregate total consumption, and aggregate equity returns.

The paper proceeds as follows. Section 2 reviews the literature, section 3 presents new
empirical findings that motivate the need for multi-frequency risk dynamics in general equi-
librium models, section 4 gives the theoretical framework, section 5 calibrates the economy
and shows that it can account for the new empirical findings, section 6 presents empirical
tests of the model, and section 7 concludes.

2 Related literature

Return Predictability. The literature on stock market return predictability is large.
Some important papers that also revisit past findings are Fama (1991), Fama and French
(1988, 1989), Cochrane (1991), and Lettau and Ludvigson (2001). The evidence is that
when prices are high, future excess returns are low. Some of the most successful forecasting
variables are the dividend/price ratio, the term spread, the short term interest rate, and
the consumption/wealth ratio. These forecasting regressions appear to have increasing ex-
planatory power over forecasting horizons up to about seven years - a fact that has been
interpreted as a consequence of very persistent time-variation in the equity risk premium.
Even though the forecasting variables are correlated with the business cycle, they are too
persistent to reflect a pure business cycle phenomenon. Importantly, the above mentioned
forecasting variables are price-related, which mechanically link them to future discount rates,
unlike the macroeconomic variables used in this paper.

There are surprisingly few studies that investigate whether traditional business cycle vari-
ables, like real GDP growth, also forecast aggregate equity returns. Along the time-series
dimension, little significant evidence of this has been provided, as far as I am aware. A re-
cent exception is Yogo (2005), who shows that the ratio of durable to non-durable aggregate
consumption, a business cycle variable, has forecasting power at the quarterly horizon. In

\footnote{The high persistence of these variables causes statistical problems in forecasting regressions, and the inferred stylized facts have therefore been the subject of critique both on statistical and economic grounds (see, e.g., Stambaugh (1999), Lewellen (2002), Campbell and Yogo (2003), Valkanov (2003) for statistical discussions, and Bossaerts and Hillion (1999), Ang and Bekaert (2001), Goyal and Welch (2004) for critical discussions on economic significance).}
the cross-sectional dimension, Chen, Roll and Ross (1986) have shown that macroeconomic variables such as industrial production are significant factors for cross-sectional return differences. Lettau and Ludvigson (2001) construct a measure of the consumption-wealth ratio by relying on the cointegrating relationship between aggregate consumption, labor income and financial assets. They show that deviations from the linear trend forecast future equity returns at frequencies that are closer to the business cycle than, for instance, that of the dividend-price ratio. While this variable provides a link to the macro economy through consumption and labor, it also contains asset prices through their measure of financial wealth.

The empirical results in this paper also imply time-varying, counter-cyclical regression coefficients on the aggregate log dividend-price and log consumption-wealth ratios. Time-varying regression coefficients in forecasting regressions have previously been considered by Engstrom (2003). He concentrates on regressions of total returns on the (not log) dividend-price ratio and does not consider interaction with business cycle variables. Menzly, Santos and Veronesi (2004) build a model with two sources of time-variation in equity risk premiums that produce time-varying regression coefficients; time-varying risk aversion and complicated cash flow dynamics. However, they do not consider forecasting regressions of aggregate market returns, but focus on individual industries, where there is stronger evidence of non-i.i.d. dividend growth. Finally, Boudoukh, Richardson and Whitelaw (1997) document nonlinearities in the relation between the equity risk premium and the slope of the term structure.

Consumption-based Models with Time-Varying Equity Premiums Heterogeneity in risk aversion and financial markets has been first formally studied by Dumas (1989) who considers two agents with different levels of relative risk aversion in a production economy and explores the implications for consumption growth and interest rates. Wang (1996) studies two agents with different relative risk aversion in an exchange economy, but also focuses on the equilibrium term structure. Both of the resulting economies are non-stationary in general, since agents have standard power utility preferences and consumption is nonstationary. Chan and Kogan (2002) solve the issue of stationarity in an economy populated by investors with different levels of relative risk aversion by letting all agents be exposed to an exogenous, economy-wide ratio habit. They focus on the price of risk and the properties of equity returns, and show how a counter-cyclical equity premium arises endogenously due to dynamic redistribution of wealth across agents. The introduction of a habit level provides a stationary notion of good and bad times through the ratio of aggregate con-
sumption to aggregate habit. Their utility specification introduces habit in such a manner that it does not affect individual agent’s relative risk aversion, following Abel (1990).

Habit formation has also been studied as a means to introduce time-varying risk aversion into preferences more directly. Campbell and Cochrane (1999) assume a difference habit in the utility function to model an agent with counter-cyclical risk aversion and high volatility of marginal utility. A difference habit specification was first used by Constantinides (1990) in a production economy in order to resolve the equity premium puzzle of Rubinstein (1976) and Mehra and Prescott (1985). Ultimately, the difference habit specification relies on very high levels of risk aversion, which both from introspection and experimental data seem unreasonable. More recent research, on the other hand, has interpreted a difference habit as a latent measure of consumption goods that exhibit significant adjustment costs. Chetty and Szieidl (2003) provide microfoundations in terms of consumption commitments for the application of an external difference habit in the utility function of the representative agent. In particular, they show that a continuum of investors in a two-good economy, where one good is subject to considerable adjustment costs, can aggregate to such a representative agent. Ait-Sahalia et al. (2004) provide empirical support for considering multiple good economies. They find that luxury good consumption is pro-cyclical and has high unconditional volatility - both features which they show help alleviate the equity premium puzzle in a two-good economy. Finally, Bansal and Yaron (2004) generate slow-moving time-variation in the equity premium by calibrating aggregate consumption growth rates to exhibit very persistent heteroskedasticity.

The theoretical model presented here combines Chan and Kogan’s economy, where agents have constant, but different, relative risk aversion, and the Campbell and Cochrane economy, where agents exhibit behavior consistent with time-varying risk aversion, due to a slow-moving difference habit. However, I interpret their difference habit as a minimum consumption level of basic goods and explicitly model a two good economy, following Ait-Sahalia et al. (2004).

In a recent paper, Calvet and Fisher (2005) investigate the impact of shocks with different persistence levels on asset prices. Using aggregate equity returns, they find empirical support that multifrequency news is important for understanding the volatility of asset prices. This research thus complements the model and empirical results presented in this paper, which also rely on multifrequency state variables.
3 Business Cycle Variables and Time-Varying Expected Returns

There is surprisingly little empirical evidence in the literature so far that traditional business cycle variables are related to time-varying equity market risk premiums. If the state of the economy matters for the equity premium, expected returns should be high in recessions and low in expansions. However, standard linear forecasting regressions using variables like real consumption- and GDP growth, or an NBER recession indicator, provide only weak evidence of such a relation. On the other hand, several financial variables like the dividend-price, the earnings-price and the consumption-price ratio do predict future excess equity returns. Since these variables are correlated with the business cycle, this has been viewed as indirect evidence that the state of the economy is a priced risk factor (see e.g. Fama and French (1989)). However, these price ratios are very persistent; much more persistent than the common (NBER) business cycle. In fact, most forecasting variables with market prices in the denominator essentially follow a generational cycle. Figure 1 gives a visual impression of the different frequencies by contrasting the aggregate dividend-price ratio with NBER business cycles. The former has experienced swings lasting about 20 years, whereas the average post-war business cycle is only about 4.5 years.

---

3 I use "cycle" loosely here, to establish intuition. It’s not clear that the price ratios, in this case the dividend yield, are really cyclical. What is important is that they are highly persistent and much more so than business cycle variables.
I argue two main points. First, since previous research has uncovered a component of the equity risk premium that is too slow-moving to reflect a pure business cycle phenomenon, we should control for this component when seeking to evaluate the relation between business cycle variables and expected excess returns. Second, a time-varying risk premium reflects time-variation in fundamental risk, investors’ risk tolerance or an interaction of the two. Fundamental risk and investors’ risk tolerance are multiplicative for the equity premium: The impact of time-varying fundamental risk on expected excess returns is increasing in the risk aversion of the marginal investor. If risk and aggregate risk tolerance vary along different cycles, the sensitivity of the equity risk premium to fluctuations in one of the cycles depends on the level of the other.

Consistent with this intuition I show that there is considerable variation in the equity risk premium at business cycle frequencies from quarterly to annual return horizons, but that the key to uncovering this evidence is to let the sensitivity of the risk premium to business cycle fluctuations vary with the level of the "generational" component of the risk premium, as measured by the dividend-price ratio, for instance.

3.1 Data

I focus on aggregate cash flows and market returns to relate the evidence to previous predictability results and general equilibrium asset pricing models. All financial data used in the empirical study are obtained from the CRSP database. The data is quarterly from 1952:Q1 - 2002:Q3 (203 observations). I chose this particular starting date, as is often done in the literature, because from this point onwards the Federal Reserve in effect has stopped pegging interest rates. In addition, reliable quarterly macro data is only readily available for the post-war period. The explanatory variables are beginning of period, while returns are end of period. All macro data are real (1996 dollars) and per capita, obtained from NIPA.

I denote log quarterly market returns in excess of the 3-month risk free rate as $r_{t+1} - r_{f,t}$. As the price-related forecasting variables I use the aggregate dividend-price ratio $(dp_t)$, consumption-price ratio $(cp_t)$ and a measure of the consumption-wealth ratio $(cay_t$, defined in Lettau and Ludvigson (2001)). These variables have all been shown to forecast excess returns, they are highly persistent, and their theoretical counterparts are important state variables in equilibrium models. I do not take a stance here on which of these variables better reflect the "true" process for the ratio of aggregate output to aggregate wealth. Instead, I consider all three as proxies that should to some extent capture the same information.
As measures of the business cycle, I use three variables based on real GDP: Last quarter’s log real GDP growth, $\Delta gdp_t$, the cyclical component of real GDP growth extracted using the Hodrick-Prescott filter on data up until time $t$ only, $HP_t$, and a time $t$ probability of the economy being in an expansion vs. a recession, $f_t$. The $f_t$ variable is the filtering probability obtained by estimating a two-state switching regime model on GDP growth. This follows Hamilton (1989), who has demonstrated that such filtering probabilities correspond very closely to the NBER definitions of business cycles. The three measures of the state of the aggregate economy should be progressively less noisy. As an additional financial variable, I include the difference between the 5 year and the 3 month U.S. treasury yields, $term_t$. Fama and French (1989) note that $term$ is increasing through every recession. They also show that it has some forecasting power for future excess stock returns. However, it differs from the other business cycle variables in that it is based on asset prices.

### 3.2 Persistence of Forecasting Variables

Table 1 gives an indication of the different persistence of the business cycle variables and the financial price ratios through the sample first-order autocorrelation of these variables at 1 quarter, 1 year, and 2 year lags. The quarterly autocorrelation of the business cycle variables ranges from 0.35 to 0.88, the annual from $-0.10$ to 0.29. At the biannual horizon all the autocorrelations are negative, with an average of $-0.07$. This is consistent with a comparatively fast moving business cycle. The financial price ratios, however, are much more persistent. The autocorrelation coefficients of both the dividend- and consumption-price ratios are very high at all frequencies, with bi-annual autocorrelations around 0.7. The $cay$ variable is less persistent, but still considerably more so than the real business cycle variables at the annual and bi-annual frequencies where it is still positive at 0.27. In sum, these financial forecasting variables are likely to pick up more persistent, "generational" components of the equity premium compared to the macroeconomic variables.

---

$4$The HP-filter is applied with a smoothing parameter $\lambda = 1600$, which is the value Hodrick and Prescott (1997) recommends for evaluating business cycle fluctuations. I use a slightly modified version that does not discard the two first and last observations, so that I obtain a time $t$ estimate of the cyclical component using data up until time $t$ only.

The filtering probabilities are estimated using the entire sample. A potential concern when we later use these in forecasting regressions is that they are subject to a "look-ahead" bias. However, the "true" time $t$ business cycle variable based on the Hodrick-Prescott filter does even better. I use the filtering probabilities in the main part of paper because of their clean economic interpretation.
Table 1: This table reports sample first-order autocorrelation from quarterly to bi-annual lags for quarterly observations of the variables used in the paper. The business cycle variables are log quarterly GDP growth, \( \Delta gdp_t = \ln(GDP_t)/\ln(GDP_{t-1}) \), a time \( t \) version of the Hodrick-Prescott filter applied to GDP growth, \( HP_t \), and an expansion vs. recession probability, \( f_t \). The financial price ratios are the long-short yield spread, \( \text{term}_t \), the log dividend yield, \( dp_t \), the log aggregate consumption (from NIPA) to market value of CRSP market index, \( cp_t \), and a measure of the consumption-wealth ratio, \( cay_t \). The sample contains 203 quarterly observations, from end of period Q4 1951 to end of period Q2 2002.

<table>
<thead>
<tr>
<th>First order Autocorrelation</th>
<th>Macroeconomic Variables</th>
<th>Financial Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \Delta gdp_t )</td>
<td>( HP_t )</td>
</tr>
<tr>
<td>One quarter</td>
<td>0.35</td>
<td>0.88</td>
</tr>
<tr>
<td>One year</td>
<td>-0.09</td>
<td>0.29</td>
</tr>
<tr>
<td>Two years</td>
<td>-0.04</td>
<td>-0.10</td>
</tr>
</tbody>
</table>

3.3 Main Empirical Results

In this section, I empirically investigate the relation between the business cycle and the equity risk premium. Panel A in table 2 shows quarterly and annual univariate excess return forecasting regression using the business cycle variables. While the negative sign on the business cycle variables (which are all high in expansions) are what we would expect, the coefficient is only significant at the 5% level at the annual forecasting horizon. Also, the adjusted \( R^2 \)'s are fairly low, but increasing in the forecasting horizon.

In contrast, panel B shows that all the financial price ratios are significant at the 5% level, both for quarterly and annual return forecasting horizons. The adjusted \( R^2 \)'s are also much higher compared to the business cycle variables. In fact, these highly persistent price ratios have been shown to forecast even longer horizon returns with a strongly increasing \( R^2 \). This has been interpreted as evidence of an equally persistent equity risk premium. The presence of such a slow-moving component in the risk premium suggests that we will have more power to uncover any business cycle variation if we control for the very slow-moving component.

Panel C of table 2 shows selected regressions using both a business cycle variable and a price ratio. In the left columns, both the dividend yield and the term spread are significant. Adding the term spread, which is the financial variable that is empirically closest to a business
cycle variable, substantially increases the adjusted $R^2$ compared to the univariate dividend yield regression. This indicates that there is considerable business cycle related variation in the risk premium that is not accounted for by the dividend-price ratio alone. This conclusion is supported in the middle columns where lagged log GDP growth has marginal forecasting power above the dividend yield. The right columns show weak evidence that log GDP growth has a marginal effect even when combined with the $cay$ variable, whose in-sample forecasting power is very large.

While there is some evidence in table 2 that business cycle variables forecast excess returns and thus relate a counter-cyclical risk premium to the business cycle, the evidence is not strong. The financial, price-related variables do much better, but seem to identify an even slower-moving component of the equity risk premium. However, it is difficult to separate different components of the risk premium using price related forecasting variables alone, since prices necessarily incorporate both effects. On the other hand, the business cycle variables reflect pure business cycle fluctuations in macroeconomic conditions.

The impact of fluctuations in the state of the aggregate economy on asset prices depends on the preferences of the marginal investor. If the marginal investor is very risk averse, asset prices will be more sensitive to changes in the level of risk than if she is more risk tolerant. At the same time, if the marginal investor is highly risk averse, prices are low and the price ratios will be high. We should therefore expect that time-variation in the risk premium due to business cycle fluctuations depends on the level of the price ratios. To investigate this intuition empirically, I run the following regression

$$r_{t+1} - r_{f,t} = \alpha_1 + \alpha_2 \text{BusCyc}_t + \beta_1 \text{PriceRatio}_t + \beta_2 \text{BusCyc}_t \ast \text{PriceRatio}_t + \varepsilon_{t+1}$$  (1)

where $\text{BusCyc}$ is a business cycle variable and $\text{PriceRatio}$ is a financial price ratio. The interaction term is what allows for time-varying sensitivity of the risk premium to business cycle fluctuations. We can construct implied regression coefficients on the business cycle variables as $\beta_{\text{BusCyc},t} = \alpha_2 + \beta_2 \text{PriceRatio}_t$. For the business cycle variables, which are high in expansions, we expect a more negative coefficient when the price ratios are high, which implies $\beta_2 < 0$. Analogously, the implied regression coefficients on the price ratios are $\beta_{\text{PriceRatio},t} = \beta_1 + \beta_2 \text{BusCyc}_t$. The conjecture $\beta_2 < 0$ means we expect the implied

5 The dividend yield, for instance, is related to future total returns on the market by an accounting identity (Campbell and Shiller (1989)).

6 Throughout the paper, I use de-meaned individual regressors. Thus, the interaction term is really $(\text{BusCyc}_t - E_T[\text{BusCyc}_t]) (\text{PriceRatio}_t - E_T[\text{PriceRatio}_t])$. 

10
### TABLE 2
Forecasting Regressions - Unconditional Approach

Table 2: The table reports estimates from OLS regression of quarterly and annual log excess stock market returns on lagged forecasting variables. The financial forecasting variables are the consumption-price ratio (cp), the dividend-price ratio (dp), a measure of the consumption-wealth ratio (cay), and the yield spread (term). The business cycle variables are quarterly log GDP growth, $\Delta gdp$, the cyclical component of GDP from a (time t) HP-filter, $HP$, and an expansion vs. recession probability, $f$ (high in expansions and low in recessions). The sample period for the forecasting variables is 1951Q4 - 2002Q2. The annual forecasting regressions are overlapping at the quarterly frequency. Newey-West corrected standard errors appear below the coefficient estimates. Independent variables are demeaned. One asterisk: significant at the 10 percent level or better. Bold font: significant at the 5 percent level or better.

#### Panel A: Real Business Cycle Forecasting Variables

<table>
<thead>
<tr>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
</tr>
</thead>
<tbody>
<tr>
<td>$const$</td>
<td>0.012</td>
<td>0.050</td>
<td>$const$</td>
<td>0.012</td>
<td>0.050</td>
<td>$const$</td>
<td>0.012</td>
<td>0.050</td>
</tr>
<tr>
<td></td>
<td>(0.006)</td>
<td>(0.020)</td>
<td></td>
<td>(0.006)</td>
<td>(0.020)</td>
<td></td>
<td>(0.006)</td>
<td>(0.020)</td>
</tr>
<tr>
<td>$\Delta gdp$</td>
<td>−0.422</td>
<td>−3.435</td>
<td>$HP$</td>
<td>−0.514</td>
<td>−1.518</td>
<td>$f$</td>
<td>−0.021</td>
<td>−0.115</td>
</tr>
<tr>
<td></td>
<td>(0.629)</td>
<td>(1.315)</td>
<td></td>
<td>(0.341)</td>
<td>(1.174)</td>
<td></td>
<td>(0.022)</td>
<td>(0.053)</td>
</tr>
<tr>
<td>$R^2_{adj}$</td>
<td>0.0%</td>
<td>3.4%</td>
<td>$R^2_{adj}$</td>
<td>0.7%</td>
<td>2.0%</td>
<td>$R^2_{adj}$</td>
<td>0.1%</td>
<td>4.0%</td>
</tr>
</tbody>
</table>

#### Panel B: Financial Forecasting Variables

<table>
<thead>
<tr>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
</tr>
</thead>
<tbody>
<tr>
<td>$const$</td>
<td>0.012</td>
<td>0.050</td>
<td>$const$</td>
<td>0.012</td>
<td>0.050</td>
<td>$const$</td>
<td>0.012</td>
<td>0.050</td>
</tr>
<tr>
<td></td>
<td>(0.006)</td>
<td>(0.020)</td>
<td></td>
<td>(0.006)</td>
<td>(0.020)</td>
<td></td>
<td>(0.006)</td>
<td>(0.017)</td>
</tr>
<tr>
<td>$cp$</td>
<td>0.035</td>
<td>0.137</td>
<td>$dp$</td>
<td>0.037</td>
<td>0.141</td>
<td>$cay$</td>
<td>2.029</td>
<td>6.828</td>
</tr>
<tr>
<td></td>
<td>(0.018)</td>
<td>(0.056)</td>
<td></td>
<td>(0.018)</td>
<td>(0.061)</td>
<td></td>
<td>(0.456)</td>
<td>(1.491)</td>
</tr>
<tr>
<td>$R^2_{adj}$</td>
<td>1.6%</td>
<td>7.8%</td>
<td>$R^2_{adj}$</td>
<td>2.3%</td>
<td>9.4%</td>
<td>$R^2_{adj}$</td>
<td>8.6%</td>
<td>24.9%</td>
</tr>
</tbody>
</table>

#### Panel C: Some Relevant Multivariate Linear Forecasting Regressions

<table>
<thead>
<tr>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
</tr>
</thead>
<tbody>
<tr>
<td>$const$</td>
<td>0.012</td>
<td>0.050</td>
<td>$const$</td>
<td>0.012</td>
<td>0.050</td>
<td>$const$</td>
<td>0.012</td>
<td>0.050</td>
</tr>
<tr>
<td></td>
<td>(0.006)</td>
<td>(0.019)</td>
<td></td>
<td>(0.006)</td>
<td>(0.020)</td>
<td></td>
<td>(0.006)</td>
<td>(0.017)</td>
</tr>
<tr>
<td>$term$</td>
<td>1.089</td>
<td>2.899</td>
<td>$\Delta gdp$</td>
<td>−0.244</td>
<td>−2.793</td>
<td>$\Delta gdp$</td>
<td>−0.053</td>
<td>−2.236*</td>
</tr>
<tr>
<td></td>
<td>(0.451)</td>
<td>(1.314)</td>
<td></td>
<td>(0.602)</td>
<td>(1.145)</td>
<td></td>
<td>(0.589)</td>
<td>(1.338)</td>
</tr>
<tr>
<td>$dp$</td>
<td>0.044</td>
<td>0.160</td>
<td>$dp$</td>
<td>0.036</td>
<td>0.132</td>
<td>$cay$</td>
<td>2.023</td>
<td>6.579</td>
</tr>
<tr>
<td></td>
<td>(0.019)</td>
<td>(0.065)</td>
<td></td>
<td>(0.018)</td>
<td>(0.062)</td>
<td></td>
<td>(0.461)</td>
<td>(1.548)</td>
</tr>
<tr>
<td>$R^2_{adj}$</td>
<td>4.7%</td>
<td>14.0%</td>
<td>$R^2_{adj}$</td>
<td>1.9%</td>
<td>11.5%</td>
<td>$R^2_{adj}$</td>
<td>8.1%</td>
<td>26.2%</td>
</tr>
</tbody>
</table>

11
regression coefficient on the price ratios to be higher in recessions.

Table 3 shows the results of running these regressions for both quarterly and annual return forecasting horizons. Each panel corresponds to a price ratio. Within each panel, I use different business cycle variables. First, log GDP growth, then the cyclical component of GDP extracted using the Hodrick-Prescott filter, and finally the filtering probability of an expansion. These variables should be increasingly less noisy measures of the business cycle. The interaction terms are negative and significant, as conjectured, and the adjusted $R^2$'s increase for all specifications relative to the univariate benchmark regressions in table 2. The statistical significance tends to increase as we move to the right within each panel, i.e. to less noisy business cycle measures, as well as when we look down across panels from $cp$ to $dp$ and $cay$, i.e. to increasingly better linear return predictors.

In both the $dp$ and the $cp$ regressions the significance and magnitude of the interaction term are increasing in the return horizon. The annual forecasting regression using the dividend yield and the time $t$ cyclical component of GDP gives the strongest result with an adjusted $R^2$ of 17.4% compared to 9.4% in the annual forecasting regression using the dividend yield only. The conditional $cay$-regressions have higher explanatory power, but display less significant interaction effects with increasing return horizons. This is because the persistence of both the $cay$ variable itself and its interaction term is lower than the corresponding for the $dp$ and $cp$ variables. For horizons longer than a year, the significance of the interaction terms decreases for all specifications (not reported).

The increase in adjusted $R^2$'s for the regressions in table 3 relative to table 2, indicate that standard return forecasting regressions using only the price ratios as the predictive variables fail to capture an economically significant portion of the variation in the equity premium at the quarterly to the annual horizon.

### 3.3.1 Evaluation of the Empirical Results

We can further interpret the results by constructing an implied regression coefficient on the price ratios as $\beta_{\text{PriceRatio},t} = \beta_1 + \beta_2 BusCyc_t$. The dividend-price ratio holds a special place, since it is mechanically related to future total returns (assuming stationarity; see Campbell and Shiller (1989)). Thus, information about next period’s excess return is there - it’s just a matter of getting to it! Figure 2 shows the implied regression coefficient on the dividend yield when the business cycle variable is the filtering probability, i.e. $\beta_{dp,t} = \beta_1 + \beta_2 (f_t - E_T [f_t])$. The regression coefficient is substantially larger in recessions than in expansions, which is
Table 3: The table reports estimates from OLS regression of quarterly and annual log excess stock market returns on lagged forecasting variables. The financial forecasting variables are the consumption-price ratio (cp), the dividend-price ratio (dp), and a measure of the consumption-wealth ratio (cay). The business cycle variables are quarterly log GDP growth, \( \Delta gdp \), the cyclical component of GDP from a (time t) HP-filter, \( HP \), and an expansion vs. recession probability, \( f \) (high in expansions and low in recessions). Data period for the forecasting variables: 1951Q4 - 2002Q2. Newey-West corrected standard errors appear below the coefficient estimates. One asterisk: significant at the 10 percent level or better. Bold: significant at the 5 percent level or better. All individual variables are demeaned, and the intercepts are not reported.

### Panel A: Price ratio = \( \log \) consumption / price (cp)

<table>
<thead>
<tr>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta gdp )</td>
<td>0.082</td>
<td>-1.945</td>
<td>( HP )</td>
<td>-0.140</td>
<td>-0.389</td>
</tr>
<tr>
<td>( (0.617) )</td>
<td>(1.414)</td>
<td></td>
<td>( (0.366) )</td>
<td>(1.154)</td>
<td></td>
</tr>
<tr>
<td>( cp )</td>
<td>0.034</td>
<td>0.126</td>
<td>( cp )</td>
<td>0.034</td>
<td>0.135</td>
</tr>
<tr>
<td>( (0.017) )</td>
<td>(0.055)</td>
<td></td>
<td>( (0.017) )</td>
<td>(0.053)</td>
<td></td>
</tr>
<tr>
<td>( \Delta gdp \times cp )</td>
<td>-2.664</td>
<td>-6.606</td>
<td>( HP \times cp )</td>
<td>-2.067</td>
<td>-5.593</td>
</tr>
<tr>
<td>( (1.973) )</td>
<td>(3.378)</td>
<td></td>
<td>( (1.046) )</td>
<td>(2.618)</td>
<td></td>
</tr>
<tr>
<td>( R^2_{adj} )</td>
<td>1.8%</td>
<td>10.9%</td>
<td>( R^2_{adj} )</td>
<td>3.5%</td>
<td>12.0%</td>
</tr>
</tbody>
</table>

### Panel B: Price ratio = \( \log \) dividends / price (dp)

<table>
<thead>
<tr>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta gdp )</td>
<td>0.111</td>
<td>-1.019</td>
<td>( HP )</td>
<td>-0.025</td>
<td>-0.035</td>
<td>( f )</td>
<td>0.005</td>
<td>-0.026</td>
</tr>
<tr>
<td>( (0.589) )</td>
<td>(1.289)</td>
<td></td>
<td>( (0.364) )</td>
<td>(1.042)</td>
<td></td>
<td>( (0.023) )</td>
<td>(0.039)</td>
<td></td>
</tr>
<tr>
<td>( dp )</td>
<td>0.035</td>
<td>0.122</td>
<td>( dp )</td>
<td>0.035</td>
<td>0.120</td>
<td>( dp )</td>
<td>0.035</td>
<td>0.123</td>
</tr>
<tr>
<td>( (0.017) )</td>
<td>(0.055)</td>
<td></td>
<td>( (0.014) )</td>
<td>(0.047)</td>
<td></td>
<td>( (0.015) )</td>
<td>(0.049)</td>
<td></td>
</tr>
<tr>
<td>( \Delta gdp \times dp )</td>
<td>-2.790</td>
<td>-9.225</td>
<td>( HP \times dp )</td>
<td>-2.075</td>
<td>-8.374</td>
<td>( f \times dp )</td>
<td>-0.116</td>
<td>-0.390</td>
</tr>
<tr>
<td>( (1.991) )</td>
<td>(3.392)</td>
<td></td>
<td>( (1.050) )</td>
<td>(2.534)</td>
<td></td>
<td>( (0.058) )</td>
<td>(0.117)</td>
<td></td>
</tr>
<tr>
<td>( R^2_{adj} )</td>
<td>3.2%</td>
<td>13.1%</td>
<td>( R^2_{adj} )</td>
<td>5.9%</td>
<td>17.4%</td>
<td>( R^2_{adj} )</td>
<td>3.8%</td>
<td>17.1%</td>
</tr>
</tbody>
</table>

### Panel C: Price ratio = \( \log \) consumption / wealth proxy (cay)

<table>
<thead>
<tr>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
<th>Return horizon</th>
<th>Quarterly</th>
<th>Annual</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta gdp )</td>
<td>0.099</td>
<td>-2.056</td>
<td>( HP )</td>
<td>-0.254</td>
<td>-1.241</td>
<td>( f )</td>
<td>0.009</td>
<td>-0.058</td>
</tr>
<tr>
<td>( (0.583) )</td>
<td>(1.347)</td>
<td></td>
<td>( (0.329) )</td>
<td>(1.051)</td>
<td></td>
<td>( (0.023) )</td>
<td>(0.057)</td>
<td></td>
</tr>
<tr>
<td>( cay )</td>
<td>1.970</td>
<td>6.516</td>
<td>( cay )</td>
<td>2.086</td>
<td>6.740</td>
<td>( cay )</td>
<td>2.247</td>
<td>6.757</td>
</tr>
<tr>
<td>( (0.438) )</td>
<td>(1.524)</td>
<td></td>
<td>( (0.472) )</td>
<td>(1.460)</td>
<td></td>
<td>( (0.470) )</td>
<td>(1.574)</td>
<td></td>
</tr>
<tr>
<td>( \Delta gdp \times cay )</td>
<td>-114.812</td>
<td>-136.155</td>
<td>( HP \times cay )</td>
<td>-63.996</td>
<td>-6.050</td>
<td>( f \times cay )</td>
<td>-5.703</td>
<td>-5.144</td>
</tr>
<tr>
<td>( (55.282) )</td>
<td>(105.682)</td>
<td></td>
<td>( (31.797) )</td>
<td>(83.910)</td>
<td></td>
<td>( (1.852) )</td>
<td>(4.215)</td>
<td></td>
</tr>
<tr>
<td>( R^2_{adj} )</td>
<td>9.6%</td>
<td>26.5%</td>
<td>( R^2_{adj} )</td>
<td>10.5%</td>
<td>25.9%</td>
<td>( R^2_{adj} )</td>
<td>12.9%</td>
<td>27.0%</td>
</tr>
</tbody>
</table>
Figure 2: Implied historical time-variation in the regression coefficient on the dividend-price ratio, when using the filtering probabilities, $f_t$, as the business cycle variable in a forecasting regression on quarterly excess returns. From estimates in table 3, $\beta_{dp} = 0.035 - 0.116(\hat{f}_t - E_T[\hat{f}_t])$. The grey bars are NBER recessions. The graph thus also shows how well the filtering probabilities estimated on log real GDP growth picks up recessions and expansions.

The time-varying regression coefficient tells us that the risk premium is much more sensitive to changes in overall investor risk tolerance (proxied by the dividend-price ratio) when times are bad. In a world where stockholders in the aggregate must bear a higher level of real risk in recessions than in expansions, asset prices will be more sensitive to changes in the risk aversion of the marginal investor when the economy is depressed.

All else equal, we expect the equity premium to be higher in recessions than in expansions. From table 3 it is clear that when the price ratios are high, the business cycle variables have the expected effect - the risk premium is significantly higher in recessions than in expansions. On the other hand, if the forecasting ratios are below their sample mean, the business cycle variable appears to have the opposite effect in some cases. However, looking at the implied regression coefficient on the business cycle variables in isolation ignores important covariance effects between the independent variables. For all the different variable combinations, the interaction term is significantly positively correlated with the business cycle variable. Thus, the prevailing effect of a recession is indeed an increase in the estimated expected returns. This is clear from figure 3, which shows the estimated risk premium from three different specifications. Panel B displays the estimated risk premium obtained regressing excess returns on the dividend-price ratio, log GDP growth and their interaction term. Recessions
(grey bars) are indeed associated with an increase in the risk premium. Also, we can see considerable more variation in the estimated risk premium at the business cycle frequency compared to Panel A, which is the estimated risk premium using the dividend-price ratio only. The business cycle fluctuations of the early fifties and mid/late seventies have the largest impact on the risk premium, and these are times when the dividend yield was high. Panel C shows the estimated risk premium when using the dividend-price ratio and the term spread as in table 2, panel C. These forecasting variables pick up much of the same business cycle variation in the risk premium as that we see in Panel B. This supports the new findings in this paper and indicates that the nonlinear interaction term is a feature of real (as in not price-based), macroeconomic forecasting variables.

In sum, the evidence presented here shows that there is substantial variation in expected excess quarterly to annual returns at the business cycle frequency. There is a statistically and economically important interaction effect between business cycle variables and future returns. In particular, the risk premium is more sensitive to macroeconomic fluctuations when a very low-frequency component of the risk premium is high. The relation has the expected sign, in that the risk premium is higher in recessions than in expansions. This evidence indicates that the risk premium has both a business cycle component and a generational component, and that accounting for both is important when relating macroeconomic activity to asset prices.

3.3.2 Robustness

I have relegated this section to the appendix, but give a summary here. I address four issues: First, the small sample properties of the t-statistics, second, whether the results arise from a particular time-period or observation, third, whether the results are due to nonlinearities in the univariate relation between the price ratios and the risk premium, and fourth, whether the results are particular to the GDP data or also related to other standard business cycle variables.

Statistical inference for forecasting regressions with persistent, endogenous regressors are plagued by small-sample problems (see, e.g., Stambaugh (1999), Campbell and Yogo (2003), Ferson and Sarkissian and Simin (2003)). The appendix presents results from a Monte Carlo experiment showing that while the regression coefficient on the price ratios are subject to the well-known Stambaugh (1999) bias, the t-statistics on the interaction terms are well-behaved in small samples. Thus, the new empirical findings in this paper of business cycle variation
Figure 3: Panel A shows estimated quarterly risk premium based on a univariate regression using the dividend-price ratio as the forecasting variable. In panel B, the risk premium forecast is based on the dividend-price ratio, lagged log real GDP growth and their interaction. Panel C shows the estimated risk premium based on the dividend-price ratio and the long-short U.S. treasury yield spread.
in the equity risk premium and the implied regression coefficients on the price ratios are robust to the usual small sample problems.

Since the business cycle variables are correlated with the price ratios, it could be that the results are due to nonlinearities in the univariate relation between the price ratios and the risk premium. Regressions adding nonlinear transformations of the price ratios on the right hand side shows that this is not the case.

Next, I show that forecasting regressions using a NBER recession indicator and lagged growth rates in Industrial Production yields similar results. I interpret this as evidence that the findings are indeed due to a business cycle factor, not an idiosyncratic GDP phenomenon. Finally, while the filtering probabilities are very intuitive and easy to interpret economically, they are estimated on the whole sample. It may therefore be that the t-statistics using this variable are inflated due to correlation between the estimation error and excess returns. However, the (business-) cyclical component of log real GDP, obtained by applying the Hodrick-Prescott (1997) filter using information up to time $t$ only, gives the same results with comparable t-statistics and $R^2_{adj}$. Since the estimation error in this case by construction is uncorrelated with future returns, this variable is not subject to the same critique. Therefore, I conclude it is unlikely that the results using $f_t$ are due to a "look-ahead" bias.

**4 The Model**

Current asset pricing models usually have only one source of time-variation in the market price of risk. To fit standard moments of asset returns, this risk factor is calibrated to operate at very low, generational frequencies and thus fails to account for the additional business cycle variation established in the previous section: We need two state variables that impact the risk premium, that interact and that operate at different frequencies.

Since aggregate consumption growth appear to be homoskedastic in the data, many models that fit the stylized facts on the joint, dynamic behavior of aggregate asset returns and consumption growth feature heterogeneous-agents or heterogeneous-goods.\(^7\) Both of

\(^7\)For examples of heterogeneous agent economies see, e.g., Chan and Kogan (2002) for heterogeneity in risk aversion, Basak and Cuoco (1999) for an economy where a subset of agents are prohibited from investing in the stock market, Constantinides, Donaldsson and Mehra (2002) for borrowing constraints on the young, Constantinides and Duffie (1996) for uninsurable income shocks. For examples of heterogeneous goods economies see, e.g., Piazzesi, Schneidel and Tuzel (2003) for housing vs. other goods, Ait-Sahalia, Parker and Yogo (2003) for luxury goods vs. basic goods, and Chetty and Szeidl (2003) for commitment goods vs. non-commitment goods. The external difference habit framework exploited by Campbell and Cochrane (1999), can be thought of as either a heterogeneous good model (see Chetty and Szeidl (2003)),
these channels naturally lend themselves to interesting risk dynamics, because shocks lead to either redistribution of wealth across different agents or redistribution of relative consumption across different goods.

In this section, I present a model with heterogeneous agents and heterogeneous goods. The presence of multiple consumption goods captures macroeconomic risks at the business cycle frequency in this model. Agents are assumed to have different levels of relative risk aversion. The average level of risk aversion in the economy is then determined by the distribution of wealth across agents (see Dumas (1989), Vanden (1999)). This distribution evolves at a generational frequency. The sensitivity of asset prices to fundamental business cycle risks is increasing in the level of average risk aversion in the economy. As an extreme example, if the marginal agent is risk neutral, variation in fundamental risks does not affect asset prices at all. Interaction effects such as those presented earlier arise naturally in this framework since the heterogeneous agent channel (the price of risk) and the heterogeneous good channel (the amount of risk) are multiplicative and operate at different frequencies.

4.1 Aggregate Endowments and Financial Securities

The focus of this model is on the implications of heterogeneity in both preferences and goods. I keep the other elements of the model as standard and as parsimonious as possible. I model a complete markets, pure exchange economy where shocks to aggregate endowments, $Y_t$, are the only source of risk. Define $y_t \equiv \ln(Y_t)$ as

$$dy_t = \mu dt + \sigma dW_t, \quad t \in [0, \infty)$$

i.e., aggregate endowments follow a geometric Brownian motion. In other words, there is one Lucas tree that yields a generic harvest $Y_t$ each moment in time. Think of this as the "basic" good. In addition, investors are assumed to have access to a technology $P$, which allows them to transform this good into a "luxury" good. $P$ is therefore the relative price of luxury goods vs. basic goods (see Eichenbaum and Hansen (1990) for a similar intuition for relative prices). The relative consumption of the two goods is endogenously determined and $Y_t = Y_t^B + PY_t^L$, where $Y_t^B$ is the equilibrium aggregate consumption of basic goods and $Y_t^L$ is the equilibrium aggregate consumption of luxury goods. Finally, there exist two long-lived securities: A riskless asset in zero net supply and a risky claim to the aggregate endowment stream in unit supply. The market is thus dynamically complete.

or a heterogeneous agent model (see Guvenen (2003)).
4.2 Agents and preferences

An equilibrium model with heterogeneous agents has implications for individual consumption patterns and portfolio choices. Empirically, the wealthy hold more stocks than the poor and their consumption growth is more volatile and more highly correlated with the stock market than the consumption growth of the poor; facts consistent with non-homothetic utility. In addition, even when controlling for the level of wealth, there is heterogeneity in portfolio holdings.\(^8\) This indicates there are differences in risk tolerance that are separate from the effects of wealth itself. I incorporate these features in my model.

I assume agents, indexed \(i\), have within-period utility (time subscripts suppressed) over luxury goods \(C^L_i\) and basic goods \(C^B_i\) as

\[
u \left( C^L_i, C^B_i; B, X, \pi, \gamma_i \right) = \frac{1}{1 - \gamma_i} \left( \frac{C^L_i}{X} \right)^{1-\gamma_i} + \frac{\pi}{1 - \gamma_i} \left( \frac{C^B_i - B}{X} \right)^{1-\gamma_i}
\]  

The utility function is non-homothetic since all agents must consume a minimum, externally given level of basic goods \(B > 0\).\(^9\) Thus, the wealthier an investor is, the larger the ratio of her luxury good consumption to basic good consumption. The distinction between luxury goods and basic goods is motivated by Ait-Sahalia et al. (2004) who show empirically that luxury good consumption (goods typically consumed only by the rich) do better at explaining asset prices than aggregate consumption. The above utility function achieves this in a general equilibrium setting: Aggregate luxury good consumption growth is a better proxy for agents’ marginal utility because it is a better measure of consumption investors can adjust at the margin. Both aggregate basic good and total consumption growth are contaminated by the unobservable consumption floor, \(B\).\(^10\)

For further intuition on the implications of the preference specification, consider agent

\(^8\)For empirical evidence on household consumption and asset holdings, see e.g. Mankiw and Zeldes (1991), Vissing-Jorgensen (2001).

\(^9\)In terms of asset prices, only the aggregate level of required basic good consumption matters in equilibrium. Thus, we could allow \(B\) to be different across agents without altering the asset pricing implications. However, the wealth distribution and agents’ optimal portfolio choice would be different.

\(^10\)The existence of a minimal level of basic goods consumption can be motivated by both a subsistence level and the presence of commitment goods. Chetty and Szeidl (2003) argue that agents pre-commit to consumption levels of certain goods, e.g. flow of housing consumption, cellular phone service, gym-membership, etc. Such commitments mean shocks to wealth will primarily be reflected in non-commitment goods that are easily adjusted at the margin. We generally think of commitment levels as endogenous, while \(B\) is assumed to be external, but Chetty and Szeidl provide a theoretical justification for this modeling choice as a reduced form representation. Another interpretation is that luxury goods are a euphemism for frictionless goods consumed mainly by the wealthy (stockholders).
i’s intratemporal first order condition

\[ C_i^L = (\pi P)^{-1/\gamma_i} (C_i^B - B) \]  

(4)

Thus, each agent’s surplus basic good consumption will move in lock-step with luxury good consumption. The minimum basic good consumption level \( B \) should be approximately constant over short intervals to be consistent with its economic interpretation. In this case, since \( B \) is a larger fraction of total consumption for the less wealthy, the volatility of total consumption growth will be higher for the rich compared to the poor. Also, the volatility of basic good consumption growth will be lower than the volatility of luxury good consumption growth. Another important implication of these preferences is that agents in equilibrium will choose luxury good consumption to be a larger fraction of total consumption in good times than in bad times. These features are discussed in greater detail below.

I follow Chan and Kogan (2002) and assume a continuum of agents, each with a potentially different curvature parameter \( \gamma_i \), drawn from a distribution \( g(\gamma) \) with support \([1, \infty)\). If an agent is very wealthy, the effect of \( B \) is small and her relative risk aversion over total wealth is close to \( \gamma_i \). The process \( X_t \) is an external habit based on past aggregate consumption (as in Abel (1990), Chan and Kogan (2002)), which I interpret as a standard of living. As shown by Chan and Kogan, a stationary \( Y_t \) makes the economy stationary in the sense that no type \( \gamma \) agent in the end controls all the wealth in the economy.\(^{11}\) In equilibrium, agents’ marginal utilities are proportional. In particular, taking two arbitrary agents, \( \gamma_i \) and \( \gamma_j \), assuming \( \gamma_i > \gamma_j \), and only considering the marginal utility of luxury goods, we have

\[ X^{\gamma_i-1} (C_i^L)^{-\gamma_i} \propto X^{\gamma_j-1} (C_j^L)^{-\gamma_j} \]

A similar condition of course holds for the basic goods consumption across agents. Holding the standard of living variable \( X \) constant, the luxury good consumption of agent \( \gamma_j \) must be more volatile compared to that of agent \( \gamma_i \). This is intuitive, since agent \( \gamma_i \) is more risk averse and thus more strongly desires a smoother consumption path. These dynamics imply that the less risk averse will hold more wealth when aggregate consumption is high relative to the standard of living \((X)\) and is the reason for the very slow-moving, counter-cyclical

\(^{11}\)Dumas (1989) and Wang (1996) show that, except for in knife-edge cases, this is not true in an economy with standard power utility agents with different levels of relative risk aversion. Relative to these economies, the standard of living process increases (decreases) the marginal utility of the less risk averse agents relative to the more risk averse when consumption levels are low (high) relative to the standard of living.
4.3 State Processes

There are two exogenous state processes in this economy, which affect all agents’ utility: The minimal consumption level $B$ and the standard of living process $X$. Both processes are specified so that the economy is stationary; both in terms of the cross-sectional distribution of wealth and the ratio of aggregate luxury good to basic good consumption.

A ratio habit as a standard of living process that affects all investors was first introduced by Abel (1990). A higher standard of living provides a complementary effect on current consumption as long as $\gamma \geq 1$. I define this process following Chan and Kogan (2002) as a geometric average of past realizations of aggregate endowments

$$x_t = x_0 e^{-(1-\theta) t} + (1-\theta) \int_0^t e^{-(1-\theta)(t-s)} y_s ds$$ (5)

where $x_t \equiv \ln(X_t)$ and $y_t \equiv \ln(Y_t)$. Note that a lower $\theta$ means higher history dependence in the standard of living process in the sense that $x_t$ in this case closely tracks recent aggregate consumption realizations. A high $\theta$ yields a slower moving standard of living process. Let $\omega_t \equiv \ln(Y_t/X_t)$ be the relative consumption. When relative consumption is high, investors are consuming relatively more than they are "used to" and marginal utility of consumption is therefore lower. Definition (5) implies that the habit is instantaneously deterministic and that relative consumption follows the stationary process

$$d\omega_t = (1-\theta) (\overline{\omega} - \omega_t) dt + \sigma dW_t$$ (6)

where $\omega_t = y_t - x_t$, $\overline{\omega} = \mu - \sigma^2/2 = 1/(1-\theta)$, and where $\mu$ and $\sigma$ are the mean and the volatility of the aggregate log consumption process given by (2). Relative consumption is stationary, mean-reverting and conditionally normally distributed with long run mean $\overline{\omega}$ and standard deviation $\sigma(\omega) = \sigma \sqrt{2/(1-\theta)}$.

The standard of living is intuitively a very slow-moving process, and I therefore calibrate the relative consumption ratio $\omega_t$ to a "generational" frequency. Note that such low frequency dynamics naturally arise if I do not introduce $X_t$. In that case, aggregate consumption $Y_t$ would take $\frac{Y_t}{X_t}$’s place as a non-stationary (i.e., even more slow-moving) state variable. The stationarity of $\frac{Y_t}{X_t}$ gives us stationarity in the wealth distribution. However, when I later
calibrate it as being close to a unit root process, it is in the spirit of standard models using power utility like Dumas (1989) and Wang (1996).

Define the aggregate minimal basic good consumption level $B$ as

$$B_t = \int B_idt$$

$B_t$ should satisfy the following properties: (1) cointegrated with total consumption, (2) bounded between 0 and $Y_t$, and (3) approximately constant over short intervals and otherwise nonnegatively, conditionally correlated with the aggregate shock. The two first conditions are partly technical: In order to achieve a stationary equilibrium, $B$ must be cointegrated with aggregate consumption. Since aggregate consumption follows a random walk with drift, $B$ cannot be constant or deterministic, but must share the same stochastic growth path. It is necessary to have $0 < B < Y$ almost surely for equilibrium to exist (no zero or negative luxury or basic good consumption). The third condition gives properties a minimal consumption level economically should exhibit. Define $B_t$ through $s_t \equiv \ln \left( \frac{Y_t - B_t}{Y_t} \right)$, where

$$ds_t = (1 - \phi) (\bar{s} - s_t) dt + \sigma \lambda (s_t) dW_t$$

and

$$\lambda (s_t) = \exp (-\bar{s}) \sqrt{1 - 2(s_t - \bar{s})} - 1.$$  

$\bar{s}$ is the steady-state level. Thus, the fraction of total consumption in excess of the minimal consumption level is a mean-reverting process with counter-cyclical conditional volatility. This specification is borrowed from Campbell and Cochrane (1999) and satisfies all of the conditions set out above. In particular, Campbell and Cochrane (1999) show that the volatility specification, $\lambda (s_t)$, satisfies the requirement that $B_t$ is constant at the steady-state, and that its innovations are nonnegatively correlated with and has lower volatility than innovations to total consumption. Since luxury good consumption growth empirically (and in this model) is pro-cyclical and has a higher unconditional volatility than total consumption growth, the counter-cyclical volatility property arises naturally: As luxury good consumption becomes a larger fraction of total consumption, its volatility must decrease to keep the volatility of total consumption constant and low. An alternative example of such volatility dynamics is given in the share-process specification in Menzly, Santos and Veronesi (2004).
4.3.1 Implications for the Dynamic Behavior of Aggregate Luxury Good and Basic Good consumption

Again, consider the intratemporal first order condition

\[ C_{i,t}^L = \left( \pi P \right)^{-1/\gamma_i} \left( C_{i,t}^B - B_t \right) \]

for an arbitrary agent \( i \). Aggregating yields

\[ Y_t^L = \int_i C_{i,t}^L di = \int_i \left( \pi P \right)^{-1/\gamma_i} \left( C_{i,t}^B - B_t \right) di \]

(9)

To obtain an analytic expression for the ratio of aggregate luxury good consumption to total consumption, I assume that

\[ \pi P = 1 \]

(10)

This assumption is not qualitatively important, but makes the exposition of the model dynamics clearer and, more importantly, allows for a clearer calibration of the parameters \( \phi \) and \( s \) from available macroeconomic data. In this case, \( Y_t^B = \int_i C_{i,t}^B di \) and \( B_t = \int_i B_{i}di \), and we have

\[ Y_t^L = Y_t^B - B_t \]

\[ = Y_t - PY_t^L - B_t \]

\[ \downarrow \]

\[ \frac{Y_t^L}{Y_t} = \frac{Y_t - B_t}{(1+P)Y_t} = \frac{e^{st}}{(1+P)} \]

(11)

Let \( y_t^L = \ln (Y_t^L) \) and \( y_t = \ln (Y_t) \). The above implies that \( ds_t = d( y_t^L - y_t ) \). Thus, the equilibrium dynamic behavior of the fraction of log luxury good consumption to total consumption follows the same stochastic process as \( s_t \). This means that luxury good consumption as a fraction of total consumption is pro-cyclical, with counter-cyclical volatility. The unconditional volatility of luxury good consumption growth is higher than that of total consumption and basic good consumption growth, as was already discussed. This all follows from the low volatility of the minimum level of basic goods consumption \( B \) (and the preferences, of course). Ait-Sahalia et al. (2004), show that their measure of aggregate luxury good consumption growth indeed is pro-cyclical with high unconditional volatility relative to aggregate consumption growth. Finally, the specification of \( s_t \) implies that there
is a maximum level of the ratio of luxury good consumption to total consumption, which is
given by
\[
\left( \frac{Y^L}{Y} \right)_{\text{max}} = \frac{e^{\sigma + \frac{1}{2}(1 - e^{\sigma})}}{1 + P}
\]
Such a maximal level is economically intuitive and its exact level is dependent on the model
calibration, which is discussed later. I will in the following refer to \( s_t \) as the "share process",
referring to its relation to \( \frac{Y^L}{Y} \).

To summarize, after having exogenously specified the minimum basic good consumption
level so as to have economically reasonable properties, the equilibrium endogenous luxury
good consumption dynamics are consistent with the available empirical evidence. The
resulting counter-cyclical volatility of luxury good consumption growth is the feature in the
model that generates higher fundamental risk in business cycle recessions compared to ex-
 pansions. Since the parameters \( \phi \) and \( \sigma \) can be calibrated using available macroeconomic
data, the introduction of a second state variable does not increase the number of free pa-
rameters (i.e., parameters inferred from asset prices) relative to relevant one state variable
benchmark models such as Campbell and Cochrane (1999) and Chan and Kogan (2002). I
will return to this when I calibrate the model in section 5.

4.4 Equilibrium
I solve for equilibrium following Wang (1996), and Chan and Kogan (2002). First, I charac-
terize the optimal consumption sharing rule. Second, I define the pricing kernel as the ratio
of marginal utilities, which implicitly defines Arrow-Debreu state prices that support this
equilibrium sharing rule. Third, optimal consumption can be obtained through sequential
trading in the risk free rate and the risky security, since these two securities complete the
market (see Duffie and Huang (1985), Wang (1996), Chan and Kogan (2002)). I solve the
complete equilibrium problem in the appendix. Here I give a partial solution that is sufficient
for pricing purposes and which conveys the intuition of the model in a clearer manner. Since
preferences are separable across goods and the relative price is constant, I consider here only
the problem of optimal allocation of luxury good consumption. Without loss of generality, I
represent all agents \( i \) with the same curvature \( \gamma_i \) as only one agent \( \gamma \). There is a continuum
of such agents with \( \gamma \in [1, \infty) \). Markets are complete and the standard of living process \( X_t \)
is external. The social planner problem is then
\[
\sup_{\{C_\gamma\}} \int_1^\infty f(\gamma) E_0 \left[ \int_0^\infty e^{-pt} \frac{1}{1-\gamma} \left( \frac{C_{\gamma,t}^L}{X_t} \right)^{1-\gamma} \, dt \right] \, d\gamma, \quad \text{s.t.} \quad \int_1^\infty C_{\gamma,t}^L \, d\gamma \leq Y_t^L, \quad \forall t \in [0, \infty)
\]

(12)

The social planner weights \( f(\gamma) \in \mathbb{R}^+ \) are assumed known. The existence of these weights implies a time \( t = 0 \) wealth distribution that supports the equilibrium. Solving this problem is equivalent to for each state and time solving

\[
\inf_{Z_t \geq 0} \sup_{C_t^L (Y_t^L, X_t; \gamma)} \int_1^\infty f(\gamma) \frac{1}{1-\gamma} \left( \frac{C_{\gamma,t}^L (Y_t^L, X_t; \gamma)}{X_t} \right)^{1-\gamma} \, d\gamma - Z_t \left[ \int_1^\infty C_{\gamma,t}^L (Y_t^L, X_t; \gamma) \, d\gamma - Y_t^L \right]
\]

(13)

where \( Z_t \) is a Lagrange multiplier over the resource constraint. It is useful to define \( \omega_t^L \equiv -\ln(1+P) + s_t + \omega_t = \ln \left( \frac{Y_t^L}{X_t} \right) \) as the log aggregate consumption of luxury goods to the standard of living. If \( \omega_t^L \) is high, there is ample amounts of luxury goods relative to the standard of living and agents with low risk aversion (\( \gamma \) low) will consume more. On the other hand, if is \( \omega_t^L \) low, agents with high risk aversion (\( \gamma \) high) will consume more. Thus, \( \omega_t^L \) determines the wealth allocation for luxury good consumption and therefore the wealth-weighted average risk aversion in the economy. The optimal sharing rule is given by the following proposition.

**Proposition 1** The optimal consumption for agent \( \gamma \) is given by

\[
C_t^* (Y_t^L, X_t; \gamma) = c_t^* (\omega_t^L; \gamma) Y_t^L
\]

and

\[
c_t^* (\omega_t^L; \gamma) = f(\gamma) \frac{1}{\gamma} e^{-\frac{1}{\gamma} z(\omega_t^L)} - \omega_t^L
\]

(14)

where \( z(\omega_t^L) \) is the log shadow price of the resource constraint over aggregate luxury consumption.

**Proof.** See appendix. \( \blacksquare \)

Note that each agent’s relative consumption level is a function of the composite state variable, \( \omega_t^L \), which is a measure of good and bad times for investors, different from the pure business cycle macro state variable \( s_t \). This variable is stationary, which means that consumption and wealth of all agents grow at the same average rate, and thus no single agent
will dominate the economy as is generally the case (see Dumas (1989) and Wang (1996)).

Since markets are complete, the stochastic discount factor is proportional to the marginal utility of an arbitrary agent $\gamma$.

**Proposition 2** The stochastic discount factor is proportional to investors’ marginal utility and can be written

$$M_t \propto \exp\left(-\rho t + z(\omega_t^{L}) + \omega_t^{L} - y_t^{L}\right)$$  \hspace{1cm} (15)

**Proof.** See appendix. ■

The variable $\omega_t^{L} = s_t + \omega_t$ is a function of two processes operating at different frequencies, and it is the interaction of $s_t$ and $\omega_t$ that makes this model highly nonlinear, as shown next.

### 4.5 Asset Prices

The price of a claim to the stream of aggregate endowments (hereafter, the stock) is

$$P_t = E_t \left[ \int_t^\infty \frac{M_s}{M_t} Y_s ds \right]$$

and we can write the price-dividend ratio as

$$\frac{P}{Y}(s_t, \omega_t^{L}) = \exp\left[-z(\omega_t^{L}) - \omega_t^{L} + s_t\right] E_t \left[ \int_t^\infty \exp\left[-\rho(s-t) + z(\omega_s^{L}) + \omega_s^{L} - s_s\right] ds \right]$$ \hspace{1cm} (16)

The price-dividend ratio is a function of the two stationary state variables in this economy. If it converges, it is therefore also stationary. Throughout the rest of the paper, I focus on parameterizations where it does converge.

**Proposition 3** The Sharpe ratio of the risky claim (the price of risk) is

$$SR(s_t, \omega_t^{L}) = \gamma_A(\omega_t^{L}) \sigma [1 + \lambda(s_t)]$$  \hspace{1cm} (17)

where $\gamma_A(\omega_t^{L})$ is the curvature of the representative agent’s utility with respect to the aggregate luxury good and is given by

$$\gamma_A(\omega_t^{L}) = -\frac{\partial z_t(\omega_t^{L})}{\partial \omega_t^{L}}$$  \hspace{1cm} (18)
The instantaneous interest rate is

\[ r_{f,t}(s_t, \omega_t) = \rho + \gamma_{A,t} \mu + \gamma_{A,t} \mu_{s,t} + (\gamma_{A,t} - 1) \mu_{\omega,t} - \frac{1}{2} \sigma^2 (1 + \lambda(s_t))^2 \left( \gamma_{A,t}^2 - \frac{\partial \gamma_{A,t}}{\partial \omega_t} \right) \]  

(19)

Proof. See Appendix □

The prevailing level of "average" risk aversion (eq. 18) is a function of the composite state variable \( \omega^L_t = \omega_t + s_t \). A positive shock to aggregate consumption leads to an increase in the aggregate supply of luxury goods relative to the standard of living, which decreases all investors’ marginal utility. However, the marginal utility of the less risk averse (low \( \gamma \)) is less sensitive to these shocks. Therefore, these investors hold riskier portfolios and, in the case of a positive shock, become relatively more wealthy than the more risk averse. Since each agent has constant relative risk aversion \( \gamma \) over luxury good gambles, an increase in \( \omega^L_t \) decreases the wealth-weighted average level of risk aversion in the economy (\( \partial \gamma_A / \partial \omega^L < 0 \)). This is analogous to the dynamics in the Chan and Kogan (2002) economy, but in their economy there is only one consumption good and the relevant state variable is \( \omega_t = \ln \left( \frac{Y_t}{X_t} \right) \).


1. In the case of homogeneous agents \( \gamma_{A,t} \) is constant and the expressions for the Sharpe ratio and, if \( \gamma = 1 \), the risk free rate are as in the Campbell and Cochrane (1999) economy. The only state variable is then \( s_t \).

2. In the case of only one good (\( \pi = 0 \)), \( s_t = 0 \) always, and the expressions for the Sharpe ratio and the risk free rate are as in the Chan and Kogan economy. The only state variable is then \( \omega_t (= \omega^L_t) \).

Proof. See Appendix □

The model presented here nests both the habit formation model of Campbell and Cochrane (1999) and the heterogeneous agent model of Chan and Kogan (2002). However, the price of risk in this economy is a function of both time-varying volatility of shocks to the real economy (luxury good consumption growth) and time-varying risk aversion. The interaction of these effects is what separates this model from the above mentioned. In particular, changes in the magnitude of risk in the real economy do not always translate into financial asset
prices in the same way. As is apparent from eq. (17), if the "average" investor is more risk averse, changes in consumption volatility have a larger effect on the equilibrium price of risk than when the "average" investor is less risk averse.

General closed form expressions for bond and stock prices do not exist in this economy. In the subsequent analysis I rely on numerical methods.

4.6 A Simple Economy - The Case of Two Agents

In this section, I present an example with closed form expressions for the discount factor. This serves to show the intuition behind the dynamics of the price of risk and the consumption sharing across agents. I next calibrate a general economy to fit unconditional moments of aggregate asset price and consumption growth dynamics.

The simplest economy that exhibits both heterogeneous goods and agents is the case of two agents with different relative risk aversion coefficients; $\gamma_i$ ($i = 1, 2$), and $\gamma_2 = 2\gamma_1$. This implies the same ratio of cautiousness parameters as in the two-agent economies considered by both Dumas (1989) and Wang (1996). The appendix shows that the optimal consumption rule in this economy is

$$C^L_2 (\omega^L_t)^* = \frac{Y^L_t}{2be^{\omega^L_t}} \left(\sqrt{1 + 4be^{\omega^L_t}} - 1\right)$$
$$C^L_1 (\omega^L_t)^* = Y^L_t - C^L_2 (\omega^L_t)^*$$

Again, the optimal consumption is a function of the composite state variable $\omega^L_t$. When luxury goods are scarce relative to the standard of living ($X_t$), the more risk averse agent ($\gamma_2$) consumes a larger share of available luxury consumption. Fixing the standard of living, the fraction of agent $\gamma_2$'s luxury good consumption to aggregate luxury good consumption goes to one as aggregate luxury goods go to zero.

$$\lim_{Y^L_t \to 0} \frac{C^L_2 (\omega^L_t)^*}{Y^L_t} = \lim_{Y^L_t \to 0} \frac{1}{\sqrt{1 + 4bY^L_t/X_t}} = 1$$

On the other hand, if times are good (i.e., $Y^L_t \to \infty$), this fraction goes to zero.

$$\lim_{Y^L_t \to \infty} \frac{C^L_2 (\omega^L_t)^*}{Y^L_t} = \lim_{Y^L_t \to \infty} \frac{1}{\sqrt{1 + 4bY^L_t/X_t}} = 0$$

Thus, the more risk averse hold more of the aggregate wealth when times are bad, which in
turn makes the "average" relative risk aversion in the economy higher. In fact

$$\gamma_{A,t}(\omega^L_t) = \gamma_2 \frac{2be^{\omega^L_t}}{4be^{\omega^L_t} + 1 - \sqrt{4be^{\omega^L_t} + 1}}$$

In figure 4, the aggregate risk aversion is plotted as a function of the composite state variable, $\omega^L_t$ (for the case $\gamma_2 = 2, \gamma_1 = 1, b = 1$), and shows graphically that $\gamma_{A,t}(\omega^L_t) \in [\gamma_1, \gamma_2]$. This corresponds to results for standard power utility agents, see e.g. Vanden (1998), which states that the aggregate relative risk aversion is bounded by the extremes of the individual agents’ relative risk aversions. The Sharpe ratio in this economy is given by

$$SR(\omega^L_t, s_t) = \gamma_A(\omega^L_t) \sigma [1 + \lambda(s_t)]$$

The price of risk ($\gamma_A(\omega^L_t)$) thus interacts multiplicatively with the amount of risk ($\sigma [1 + \lambda(s_t)]$). In the corresponding homogeneous agent economy (analogous to Campbell and Cochrane (1999)), the price of risk is constant and the Sharpe ratio is

$$SR_{CC}(s_t) = \gamma_A \sigma [1 + \lambda(s_t)]$$

The dynamic behavior of the price of risk is, in the latter case, only due to changing relative consumption across the two goods. On the other hand, in a two agent economy with only one good (analogous to Chan and Kogan (2002), $\omega^L_t = \omega_t$), the amount of risk is constant and the Sharpe ratio is

$$SR_{CK}(\omega_t) = \gamma_A(\omega_t) \sigma$$
The dynamic behavior in the last case arises because of dynamic redistribution of wealth between the two agents, which in turn yields time-varying aggregate relative risk aversion. In other words, the dynamic behavior of the price of risk in these benchmark models line up along only one dimension. In the model proposed here, there are two sources of time-varying price of risk - macroeconomic risk and aggregate risk aversion - that interacts multiplicatively.

5 Simulations and Calibration

In this section, I calibrate the general, continuum of agents version of the model to standard moments of aggregate consumption growth, aggregate luxury good consumption growth and asset returns.

5.1 Calibration

I calibrate the model at the quarterly frequency, but report annualized parameter values where appropriate. As explained in the beginning of the previous section, the dynamic behavior of the share process, $s_t$, is governed by the autocorrelation parameter $\phi$ and the average level $\bar{s}$. I set $\phi = 0.75^4 = 0.32$, to replicate the quarterly autocorrelation of business cycle fluctuations, as measured by the filtering probabilities. Ait-Sahalia et. al. (2004) have compiled a unique data set on aggregate luxury good consumption, which covers 41 years (1961 - 2001). The estimated standard deviation of annual real log luxury good consumption growth rates from this data is 9.5%. Since the value of $\bar{s}$ directly determines the volatility process of aggregate log luxury good growth in the model, I calibrate this parameter to fit the standard deviation estimated above. This is achieved at $\bar{s} = \ln(0.365)$. I do not here attempt to further define what exactly constitutes "luxury goods". Their data include high-end wine consumption, restaurant visits, jewelry and clothes. Some of these items have durable aspects (at least to the poorer of us) but, on the other hand, high-end fashions are fairly short-lived and thus the classification of this as nondurable may well be appropriate for the rich. The implicit assumption is that the empirical growth rates from this data are proportional to "true" luxury good consumption.

\footnote{I thank Motohiro Yogo for making the data available on his web site: http://finance.wharton.upenn.edu/~yogo/}
Following Chan and Kogan (2002), I assume the social planner weights

\[ f(\gamma) = (\gamma - 1) \exp\left(-a_1 \gamma - a_2 \gamma^2\right) \]  

(20)

In the calibration, \( a_1 = 5.50 \) and \( a_2 = 0.028 \), and the corresponding \( f(\gamma) \) is given in figure 5. The weights are centered around agents with utility curvature over luxury good consumption \( \gamma \) between 1 and 3. However, this does not translate directly to the relative risk aversion over total wealth, which I discuss below.

The chosen parameter values are given in table 4. I calibrate the model to the aggregate moments of the 1889 - 1994 sample. The higher volatility of both the risk free rate and of aggregate consumption growth in the longer sample makes it easier for the model to match all the unconditional moments. In particular, higher consumption volatility means the relative consumption process, \( \omega_t \), has higher volatility, which in turn yields more interesting dynamics in the time-variation of the level of aggregate risk aversion. Remember, it is the composite variable \( \omega^L_t = s_t + \omega_t \) that governs the level of aggregate relative risk aversion, and if the share process, \( s_t \), moves a lot more than \( \omega_t \) the dynamics of \( \omega^L_t \) are swamped by \( s_t \). In addition, I cannot match the low volatility of post-war risk free rates with the low post-war volatility of aggregate consumption and the specifications chosen for \( s_t \) and \( \omega_t \).

While the share process \( s_t \) represents changing fundamental risk (the conditional volatility of luxury good consumption growth) at the business cycle frequency, the standard of living process \( X_t \) is parameterized such that log relative consumption, \( \omega_t \), has an annual autocorrelation coefficient, \( \theta \), of 0.95. This makes the autocorrelation of the dividend yield close to what we find in the data and is what causes the risk aversion of the marginal agent.
Table 4
Parametrization

Table 4: Parametrization. All parameters are expressed in annual terms.

<table>
<thead>
<tr>
<th>&quot;Real&quot; parameters</th>
<th>&quot;Financial&quot; parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.0172</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.0332</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.32</td>
</tr>
<tr>
<td>$\bar{\pi}$</td>
<td>$\ln(0.365)$</td>
</tr>
<tr>
<td>$a_1$</td>
<td>5.50</td>
</tr>
<tr>
<td>$a_2$</td>
<td>0.028</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.95</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.059</td>
</tr>
</tbody>
</table>

to be slowly time-varying at a generational frequency.

Table 5 shows relevant annual moments from the calibrated model and historical data. The parameters are chosen to match the first two moments of aggregate consumption growth and the real risk free rate, the market Sharpe ratio, luxury good consumption growth volatility, and the first order autocorrelation of business cycle fluctuations as measured by the quarterly filtering probabilities. The model is able to match these moments well. Moments that are not matched in the calibration (labelled "free" moments) are given on the right half of table. Overall the model does a good job of fitting these moments also. In particular, the equity risk premium and volatility are very close to the historical values. The model does not quite match the high autocorrelation of the dividend yield. This is due to the business cycle component of expected returns: The persistence of the dividend yield is a weighted average of the persistence of the two state variables. A higher persistence of the standard of living process, $X$, or counter-(business) cyclical expected dividend growth rates could remedy this. The autocorrelation of the dividend yield is, however, considerably more persistent than the business cycle variable, $s_t$. Finally, the average level of luxury good consumption to total consumption is not determined as we have not set a value for the relative price, $P$; remember, $\frac{Y_t^L}{Y_t} = \frac{e^{s_t}}{1+P}$. Since $P > 0$ and $e^{\bar{\pi}} = 0.365$, $E \left[ \frac{PY_t^L}{Y_t} \right] \in (0, 0.365]$, which arguably is a reasonable interval for the true, real-world mean of the fraction of total wealth spent on luxury goods in the economy to reside in. This magnitude could be further determined by estimating $P$ from available data, but this exercise would add little to the intuition and focus of the model.
Table 5: Moments - Matched and "Free"

Table 5: Moments - Matched and "free" from 100,000 quarterly simulations. All parameters are expressed in annual terms. Historical values taken from Chan and Kogan (2002). Returns are in percentages. Statistics based on returns are in percentages. A dagger implies the moment is estimated on a different sample: The annual autocorrelation of the log price-dividend ratio comes from the dataset used in the empirical section. The volatility of log growth in aggregate luxury good consumption is estimated from data compiled by Ait-Sahalia, Parker and Yogo (2004).

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Historical data 1889 - 1994</th>
<th>Simulated Our Model</th>
<th>Historical data 1889 - 1994</th>
<th>Simulated Our Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;Matched&quot; moments</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E(\Delta y)$</td>
<td>1.72</td>
<td>1.72</td>
<td>4.18</td>
<td>4.74</td>
</tr>
<tr>
<td>$\sigma(\Delta y)$</td>
<td>3.32</td>
<td>3.32</td>
<td>17.74</td>
<td>19.51</td>
</tr>
<tr>
<td>$E(r_f)$</td>
<td>2.92</td>
<td>2.92</td>
<td>22.48</td>
<td>18.24</td>
</tr>
<tr>
<td>$\sigma(r_f)$</td>
<td>3.00</td>
<td>3.36</td>
<td>0.28</td>
<td>0.19</td>
</tr>
<tr>
<td>$E(r_f)/\sigma(r_f)$</td>
<td>0.24</td>
<td>0.24</td>
<td>0.90†</td>
<td>0.72</td>
</tr>
<tr>
<td>$Autocorr(s_t)$</td>
<td>0.32†</td>
<td>0.32</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma(\Delta y_L)$</td>
<td>9.5†</td>
<td>9.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

"Free" moments

5.2 The Risk Aversion of the Very Wealthy

In this economy the less wealthy have large risk aversion over wealth since the minimum basic consumption level is a relatively large fraction of their total consumption. However, the effect of the minimum basic good consumption level on an agent’s risk aversion decreases the wealthier the investor is per the discussion in section 4.2. Accordingly the volatility of the total consumption growth of the wealthiest approaches the volatility of their luxury good consumption growth. A reasonable distribution of the utility curvature parameter $\gamma$ should yield coefficients of relative risk aversion over total consumption for the very wealthy comparable to what has been found in available micro-data. Figure 6 shows the simulated path of 203 quarterly observations (as in the sample in the empirical section) of the relative risk aversion of the representative agent over "luxury good gambles", $\gamma_A(\omega^L_t)$.

In the simulated sample, the level lies between about 2.5 and 4.5. We can clearly see the effect of the composite state variable $\exp(s_t + \omega_t) = \frac{Y^L_t}{X^T_t}$ on the dynamics of aggregate risk aversion. There is a long term cycle, induced through $\omega_t$, and shorter cycles induced by $s_t$. When the relative risk aversion is higher, more risk averse agents controls more of the aggregate wealth and vice versa. These levels of risk aversion are well in line with estimates.
of the relative risk aversion of the wealthiest in disaggregate data.

5.3 The Dynamic Behavior of the Price of Risk and the Equity Risk Premium

Figure 7 shows conditional aggregate asset moments as a function of the two state variables in this economy; the share process $S_t$ (the business cycle variable), and the relative consumption $\omega_t$ (the generational variable). I label a low $S$ a "recession", and a high $S$ an "expansion", which correspond to a low and high fraction of luxury good consumption to aggregate consumption, respectively. When $\omega_t$ is high, i.e. consumption is high relative to the standard of living, the risk aversion of the marginal agent is low, and vice versa.

The upper half of figure 7 shows that the Sharpe ratio and the equity risk premium are counter-cyclical, for both kind of cycles. Importantly, the sensitivity of these moments to changes in the business cycle variable, $S_t = e^{s_t}$, is higher when the risk aversion of the marginal agent is otherwise high, i.e. when $\omega_t$ is low (the slope is steeper). Thus, the business cycle variable has larger impact on the risk premium when the level of aggregate relative risk aversion is high. This shows that the model generates dynamic behavior in both the market price of risk and the risk premium consistent with the intuition that changes in macro economic risks matter more for aggregate pricing when the marginal investor is more risk averse. The lower half of figure 7 shows the conditional moments of equity volatility.
and the price-dividend ratio.\textsuperscript{13} The model also generates interesting patterns of countercyclical volatility in equity returns at both frequencies. The risky asset is a claim to total consumption and consumption growth rates are i.i.d. The time-variation in return volatility is therefore solely due to discount rate effects, which in this model are present both at the business cycle and at a generational cycle. The price-dividend ratio is of the same reason pro-cyclical at both frequencies.

The model presented here thus displays a counter-cyclical equity premium, volatility and Sharpe ratio, which benchmark models such as Campbell and Cochrane (1999), Chan and Kogan (2002), and Bansal and Yaron (2004) also do. However, it exhibits these dynamics at a standard business cycle frequency in addition to a generational frequency.

5.4 The Price-Dividend Ratio and the Business Cycle

Figure 1 in section 3 gave a visual impression of the different frequency of the price-dividend ratio and NBER business cycles over the last 50 years. Anecdotally, this graph shows that the dividend-price ratio empirically tends to increase through recessions, but that its overall level is not directly linked to business cycle fluctuations. Rather it appears to follow a more persistent process. Second, business cycle recessions are shorter than expansions. In the following, I show that the multifrequency model presented in this paper can explain and replicate these dynamics.

Figure 8 shows a simulated time-series plot of 203 quarterly observations of the log dividend-price ratio ($dp$), and the share process ($s$). The grey, vertical bars correspond to "recessions" (defined as periods lasting longer than 1 quarter where $s < \bar{s}$). First, the dividend yield is increasing throughout a recession, as it does empirically (see figure 1). Second, the level of the dividend yield itself is not a one to one indicator of whether the economy is experiencing a recession or expansion. As an example, the level of the dividend yield at point "A" is lower than at point "B" even though the former is a recession and the latter an expansion. Third, recessions are on average shorter than expansion - a feature due to the asymmetric volatility of $s$: The volatility of luxury good consumption growth is higher in recessions than in expansions.

\textsuperscript{13}Equivalently, the wealth to consumption ratio, since consumption equals dividends in this model.
Figure 7: A dashed line means aggregate risk aversion is high given a level of the business cycle variable $s_t$, i.e. $\omega_t$ is low. The solid line shows the case where aggregate risk aversion is low; $\omega_t$ is high. The mean of $S = \exp(s)$ is 0.365. The equity risk premium and Sharpe ratio and volatility are all counter-cyclical; both with respect to the business cycle variable ($S$) and the even slower-moving aggregate risk aversion. The Price-Dividend ratio is pro-cyclical at both frequencies.
5.5 Implications for Return Forecasting Regressions

As evident from figure 7, the risk premium is a function of both business cycle fluctuations as well as the generational fluctuations in the risk aversion of the marginal agent. But, how much does each component contribute to its overall volatility? Panel A in table 6 decomposes the equity premium by regressing the quarterly expected excess equity returns on both the business cycle variable, $S$, and the log dividend yield, $dp$. The data is simulated from the calibrated model, and the coefficients are population values. Specification 1 shows that the business cycle variable can explain 67.3% of the variance in the quarterly equity premium. The sign is negative, as expected. The dividend yield, which is a function of both state variables, can explain 71.3%. Specifications 3 and 4 are the most interesting ones: While adding the business cycle variable in a linear fashion only has a slight marginal impact, the interaction of $S$ and $dp$ is increases the explanatory power from 72.6% to 95.3%. This reflects the impact of the multiplicative interaction of the risk aversion of the marginal agent (heterogeneous agents) and the real risk (heterogeneous goods) on the equity premium. As expected, the sign on the interaction term is negative: The impact of business cycle fluctuations is larger (more negative) when the dividend yield is high, i.e. the marginal agent is more risk averse. These regressions are ad hoc in the sense that the economy does not allow an analytic characterization of the true relation between these variables and the conditional equity premium. But at an $R^2$ of 95.3%, the final specification is capturing
almost all of its variation.

To test the model’s time-series implications, we need to rely on empirical proxies for $S$. Panel B of table 6 shows the results of using log aggregate luxury good consumption growth and the cyclical component of aggregate consumption, extracting using the Hodrick-Prescott filter with smoothing parameter $\lambda = 1600$, as recommended by these authors for capturing business cycle phenomena, on data up until time $t$, only. This filter is a standard method to extract business cycle dynamics from macroeconomic time series. In both regressions, the business cycle proxy has a negative coefficient, the dividend-price ratio a positive coefficient, and importantly the coefficient on the interaction term is negative. The $R^2$ is 84.0% and 83.2%, respectively. Thus, while the specifications are not able to pick up all the variation of the equity premium, they can explain a large fraction and more than the dividend-price ratio alone. These regressions on simulated data serve as a justification for interpreting the forecasting regressions of the next section as evidence supportive of the model’s main predictions.

6 Empirical Tests

In this section, I verify the main time-series predictions of the multifrequency model in available data. The evidence presented in section 3 motivated a model with two sources of time-variation in the price of risk that i) operate at a generational and a business cycle frequency, respectively, and that ii) leads to interesting interaction effects between real and financial measures of the two cycles. However, that evidence was related to real GDP growth, which does not have a clear counter-part in an exchange economy. Therefore, tests of the model proposed here rely on consumption-based measures of the business cycle.

First, I turn to the empirical properties of luxury good consumption. In the model, aggregate luxury good consumption growth is pro-cyclical with counter-cyclical conditional volatility. Table 7 verifies that an empirical measure of luxury good consumption growth indeed displays these dynamics. The luxury good consumption data is compiled by Ait-Sahalia, Parker and Yogo (2004), it is based on sales data and consists of 41 annual real growth observations for the years 1961 - 2001. Specification 1 and 2 in Panel A verify that the annual, real log luxury good consumption growth indeed is significantly and positively related to contemporaneous, real log consumption and GDP growth. Thus, luxury good consumption is a pro-cyclical variable. The specification for the share process, $s_t$, also implies that the volatility of luxury good consumption growth is higher when aggregate consumption growth
Table 6: Decomposing the Model’s Equity Premium Dynamics

Table 6: Panel A of this table reports population coefficients when regressing the quarterly equity risk premium on the business cycle factor, \( S \), the log dividend/price ratio, \( dp \), and their interaction term using simulated data from the model. Both \( S \) and \( dp \) have been demeaned. The \( R^2 \)'s thus say how much of the model-implied variation in the equity risk premium is explained by each specification. Panel B reports population coefficients when regressing the quarterly equity risk premium on empirical proxies of the business cycle factor, \( \Delta y^L \) and \( y_{HP} \), the log dividend/price ratio, \( dp \), and their interaction term using simulated data from the model. \( \Delta y^L \) is the quarterly log luxury good consumption growth, \( y_{HP} \) is the cyclical component of aggregate consumption obtained using the Hodrick-Prescott filter on quarterly log consumption with smoothing parameter \( \lambda = 1600 \) as recommended by Hodrick and Prescott (1996) using data up to time \( t \), only. This is a business-cycle frequency variable. \( y_{HP} \), \( \Delta y^L \) and \( dp \) have been demeaned. Again, the \( R^2 \)'s say how much of the model implied variation in the equity risk premium is explained by each specification.

**Panel A: Quarterly Risk Premium Decomposition**

<table>
<thead>
<tr>
<th>Specification</th>
<th>intercept</th>
<th>( S )</th>
<th>( dp )</th>
<th>( S \ast dp )</th>
<th>( R^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.010</td>
<td>−0.430</td>
<td></td>
<td></td>
<td>67.3%</td>
</tr>
<tr>
<td>2</td>
<td>0.010</td>
<td></td>
<td>0.042</td>
<td></td>
<td>71.3%</td>
</tr>
<tr>
<td>3</td>
<td>0.010</td>
<td>−0.153</td>
<td>0.029</td>
<td></td>
<td>72.6%</td>
</tr>
<tr>
<td>4</td>
<td>0.010</td>
<td>−0.276</td>
<td>0.011</td>
<td>−0.962</td>
<td>95.3%</td>
</tr>
</tbody>
</table>

**Panel B: Quarterly Forecasting Regressions with Empirical Proxies**

<table>
<thead>
<tr>
<th>Specification</th>
<th>intercept</th>
<th>( y_{HP} )</th>
<th>( dp )</th>
<th>( y_{HP} \ast dp )</th>
<th>( R^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.010</td>
<td>−0.0411</td>
<td>0.0351</td>
<td>−0.6161</td>
<td>84.0%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Specification</th>
<th>intercept</th>
<th>( \Delta y^L )</th>
<th>( dp )</th>
<th>( \Delta y^L \ast dp )</th>
<th>( R^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0.010</td>
<td>−0.0112</td>
<td>0.0368</td>
<td>−0.1813</td>
<td>83.2%</td>
</tr>
</tbody>
</table>
TABLE 7
The Dynamic Behavior of Luxury Good Consumption

Table 7: Panel A reports estimates from OLS regression of annual log luxury good consumption growth on contemporaneous annual log aggregate consumption and GDP growth. White standard errors appear below the coefficient estimates. One asterisk means significant at the 10 percent level, while bold font means significant at the 5 percent level or better. Panel B reports coefficients from an OLS regression of the demeaned, squared luxury good consumption growth on contemporaneous aggregate log consumption growth. The sample here is annual from 1961 - 2001, which yields 41 observations.

<table>
<thead>
<tr>
<th>Panel A: The Pro-cyclicality of Luxury Good Consumption Growth: $\Delta y_t^L$</th>
<th>Panel B: The Volatility of Luxury Good Consumption Growth: $(\Delta y_t^L - \bar{\Delta y_t^L})^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>#</td>
<td>intercept</td>
</tr>
<tr>
<td>1</td>
<td>0.011</td>
</tr>
<tr>
<td>(0.022)</td>
<td>(0.620)</td>
</tr>
<tr>
<td>2</td>
<td>0.039*</td>
</tr>
<tr>
<td>(0.022)</td>
<td>(0.644)</td>
</tr>
</tbody>
</table>

over the year is low, i.e., when the aggregate economy is depressed. With only 41 annual observations we have very low power to uncover time-varying volatility. Specification 3 in Panel B of table 7 shows that luxury good consumption growth squared is significantly and negatively related to the aggregate consumption shock. In other words, a negative shock to aggregate consumption gives a stronger response in luxury good consumption than a positive shock, consistent with the model assumption of counter-cyclical volatility. The latter was verified by running the same regression on simulated, annual data from the calibrated model. I conclude that the model implied dynamic behavior of luxury good consumption growth is consistent with the available empirical evidence.

Next, I turn to the model’s implication for the equity premium. Panel A of table 8 shows a regression of annual excess equity market returns on lagged annual, aggregate luxury good consumption growth, the lagged log dividend-price ratio and their interaction term, as in panel B of table 6. Specification 2 shows that the interaction term is negative and highly significant, while the coefficient on luxury good consumption growth is negative, but not significant. Thus, the signs are as predicted and the regressions’ explanatory power is increased relative to the univariate regression using the lagged dividend-price ratio, only
Next, panel B shows forecasting regressions which test the model implications on a longer sample with both quarterly and annual data. These rely on the cyclical component of aggregate consumption growth as the business cycle measure. This measure is constructed using the Hodrick-Prescott filter on data up until time $t$, only, and is thus a valid forecasting variable. To initiate the filter, I use real consumption data from 1947Q1. However, the sample used in the forecasting regressions corresponds exactly to the sample in section 3, (1952Q1 – 2002Q3). The annual return regressions are overlapping at a quarterly frequency, and each regression has 203 observations. Again the coefficient on the dividend-price ratio and the business cycle variable interaction term is negative, highly significant and increase the regressions’ explanatory power as measured by the adjusted $R^2$ relative to the univariate dividend yield forecasting regression (see table 2).

These regressions thus explicitly link the dynamics of a measure of aggregate luxury good consumption growth and a measure of the business cycle component of aggregate consumption to future excess stock market returns in a manner consistent with the model’s main predictions. More generally, the evidence presented here support consumption-based explanations for fluctuations in the equity premium at the business cycle frequency, separate from previous models which have focused on consumption-based state variables that operate at a generational frequency.

## 7 Conclusion

I have presented and analyzed an economy with heterogeneous agents and heterogeneous goods. Investors have different levels of relative risk aversion and consume both luxury goods and basic goods. The volatility of luxury good consumption growth is counter-cyclical and is the source of changing risks over the business cycle. The impact of fundamental risks on asset prices depends on the risk aversion of the marginal agent. Heterogeneity in risk preferences implies that less risk averse agents, who hold a smaller weight of risky assets in their portfolios, will hold more of the aggregate wealth when times are bad. The business cycle is only one measure of bad vs. good times. In addition, there is a very slow-moving standard of living process which affects agents’ marginal utility of consumption. When aggregate consumptions is low relative to the standard of living, the more risk averse agents will hold more of the aggregate wealth since their marginal utility is more sensitive to adverse changes in relative consumption. In equilibrium, these sources of heterogeneity
TABLE 8
Annual Forecasting Regressions Using Consumption-Based Forecasting Variables

Table 8: Panel A reports estimates from OLS regression of annual log excess stock market returns on lagged annual log luxury good consumption growth, the dividend price ratio and their interaction. The data is from 1961 - 2001 for the forecasting variables. Both luxcons and dp (the independent variables) are demeaned. Panel B reports coefficients from an OLS regression of both quarterly and annual excess returns on the lagged dividend-price ratio, the cyclical component of aggregate consumption extracted using the Hodrick-Prescott filter with data up until time $t$, only, and their interaction. The sample here is 1951Q4 - 2002Q2 for the forecasting variables. White standard errors appear below the coefficient estimates. One asterisk means significant at the 10 percent level, while bold font means significant at the 5 percent level or better.

### Panel A: Annual Forecasting Regressions - Luxury Good Consumption

<table>
<thead>
<tr>
<th>Specification</th>
<th>intercept</th>
<th>$\Delta y_t^L$</th>
<th>dp</th>
<th>$\Delta y_t^L dp$</th>
<th>$R_{adj}^2$</th>
<th># obs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.032</td>
<td>0.111*</td>
<td>0.111</td>
<td>4.4%</td>
<td>41</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.024)</td>
<td>(0.063)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.025</td>
<td>−0.184</td>
<td>0.121</td>
<td>−1.206</td>
<td>6.6%</td>
<td>41</td>
</tr>
<tr>
<td></td>
<td>(0.023)</td>
<td>(0.196)</td>
<td>(0.058)</td>
<td>(0.533)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Panel B: Quarterly and Annual Forecasting Regressions - Aggregate Consumption

<table>
<thead>
<tr>
<th>Specification</th>
<th>intercept</th>
<th>$y_{HP,t}$</th>
<th>dp</th>
<th>$y_{HP,t} \times dp$</th>
<th>$R_{adj}^2$</th>
<th># obs</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.010</td>
<td>0.209</td>
<td>0.042</td>
<td>−3.779</td>
<td>3.6%</td>
<td>203</td>
</tr>
<tr>
<td>Quarterly</td>
<td>(0.006)</td>
<td>(0.620)</td>
<td>(0.016)</td>
<td>(1.714)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.045</td>
<td>0.737</td>
<td>0.156</td>
<td>−10.759</td>
<td>12.9%</td>
<td>203</td>
</tr>
<tr>
<td>Annual</td>
<td>(0.018)</td>
<td>(1.656)</td>
<td>(0.049)</td>
<td>(4.200)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
imply an equity risk premium which has both a business cycle and a very slow-moving, generational component. The two components interact, as the required risk premium is a multiplicative function of the risk aversion of the marginal investor and the amount of risk this agent must bear. In particular, if the marginal agent is relatively risk averse, the price of risk is high and therefore changes in the amount of risk matter more for asset prices. I confirm the model’s main time-series prediction for the equity premium in aggregate return forecasting regressions with consumption-based business cycle variables, the dividend-price ratio and their interaction term as the predictive variables.

The motivation for considering two sources of time-variation in the aggregate equity premium is two-fold. First, current asset pricing models that fit the joint dynamics of aggregate consumption growth and asset returns identify a component of the risk premium operating at frequencies substantially below standard business cycle fluctuations. Yet the intuition provided usually rests on a business cycle story, which says that the equity risk premium is high in recessions and low in expansions. Second, I present new evidence that there is a standard business cycle component of the equity premium. This component is related to GDP growth and interacts with proxies for the previously identified slower-moving component in a nonlinear fashion. The evidence is consistent with intuition, as recessions are associated with a higher conditional equity premium. However, the impact of business cycle fluctuations on the equity premium is higher when the risk premium is otherwise high, as measured by highly persistent, generational forecasting variables like the dividend-price ratio.

More generally, a separation of equity premium dynamics at a standard business cycle frequency and at a longer run frequency may be important when trying to match macroeconomic models of the business cycle to equity market prices. The evidence presented in this paper show that one ought not attempt to force time-variation in the equity premium in such a model to follow standard business cycle fluctuations only, but rather allow for a slower-moving component which interacts with the business cycle dynamics.

I conclude that the model proposed here can account for the empirical findings. It provides a novel economic explanation for the observed phenomena, and is the first, as far as I know, to formalize the interaction of time-varying macroeconomic risks at the business cycle frequency and time-variation in aggregate risk preferences at both the business cycle and a generational frequency.
References


[33] Lettau and Ludvigson (2001), "Resurrecting the (C)CAPM: A Cross-Sectional Test when Risk Premia are Time-Varying", *Journal of Political Economy* 109, 1238 - 1287


[41] Piazzesi, Schneider and Tuzel (2003), "Housing, consumption and asset pricing", working paper, GSB University of Chicago

46


8 Appendix

8.1 Robustness of the Forecasting Regression Results

This section evaluates the robustness of the empirical results concerning forecasting with real business cycle variables along four dimensions. First, the small sample properties of the t-statistics, second, whether the results arise from a particular time-period or observation, third, whether the results are due to nonlinearities in the univariate relation between the price ratios and the risk premium, and fourth, whether the results are particular to the GDP data or related to other typical business cycle variables.

8.1.1 Small-sample Properties of t-statistics

I evaluate the small sample properties of t-statistics from predictability regressions using the dividend yield and the quarterly log real GDP growth as forecasting variables. Under the null hypothesis of no predictability, I assume the system can be represented as

\[ r_{t+1} - r_f,t = \varepsilon_{r,t+1} \]
\[ dp_{t+1} = \phi dp_t + \varepsilon_{dp,t+1} \]
\[ gdp_{t+1} = \varphi gdp_t + \varepsilon_{gdp,t+1} \]

where, without loss of generality, all variables have been demeaned. The innovations \( \varepsilon = [\varepsilon_{r,t+1} \varepsilon_{dp,t+1} \varepsilon_{gdp,t+1}]^T \sim N(0, \Sigma) \), where \( \Sigma \) is the covariance matrix for the contemporaneous shocks. Estimating this system by OLS yields \( \phi = 0.98 \) and \( \varphi = 0.35 \), highlighting the large difference in persistence of the independent variables. The estimated correlation matrix and standard deviations of the shocks are

\[
\text{CorrMat} \begin{pmatrix}
\varepsilon_r \\
\varepsilon_{dp} \\
\varepsilon_{gdp}
\end{pmatrix} = \begin{bmatrix}
1 & -0.966 & 0.061 \\
-0.966 & 1 & -0.060 \\
0.061 & -0.060 & 1
\end{bmatrix}
\]

\[
\text{StdDev} \begin{pmatrix}
\varepsilon_r \\
\varepsilon_{dp} \\
\varepsilon_{gdp}
\end{pmatrix} = \begin{bmatrix}
0.0828 \\
0.0774 \\
0.0091
\end{bmatrix}
\]

As usual, the contemporaneous correlation between shocks to the dividend yield and excess returns is very high and negative. This coupled with the strong persistence of the dividend yield is the
reason for the well-known Stambaugh (1999) bias. However, the correlations concerning innovations to log real GDP growth are close to zero and statistically insignificant (not reported). I run 30,000 Monte-Carlo simulations on this system of samples of 203 observations, as in the data used in this study, for the predictability regression

\[ r_{t+1} - r_{f,t} = \alpha_0 + \alpha_1 (gdp_t - E_T [gdp_t]) + \beta_0 (dp_t - E_T [dp_t]) + \beta_1 (gdp_t - E_T [gdp_t])(dp_t - E_T [dp_t]) + \epsilon_{t+1} \]  

(24)

Here \( E_T [\cdot] \) denotes the sample mean over the \( T = 203 \) samples. I am interested in the t-statistics on the individual regression coefficients.

Figure 9 show the Monte-Carlo distribution of the t-statistics on \( \beta_0 \) and \( \beta_1 \) in panels A and B, respectively. Panel A shows that the t-statistic on \( \beta_0 \) is upwards biased, as is expected from Stambaugh (1999). Evidence of (generational) predictability of excess returns using the dividend yield and the econometric challenges the properties of this regressor pose are extensively discussed in Valkanov (2004), Campbell and Yogo (2004), among others. The empirical contribution in this paper is to document time-variation in expected returns at the business cycle frequency and therefore the t-statistic on \( \beta_1 \) is of special interest. Panel B shows that it has good small sample properties - it appears unbiased and has correct size. Empirically, the interaction term has relatively low persistence (around 0.3 in this case) and its innovations have low correlation with excess returns. These two factors are the culprits for the poor small sample performance of the t-statistic on the dividend yield. The Monte-Carlo experiment also shows that the t-statistic on the linear business cycle regressor, \( \alpha_1 \), is equally well-behaved in small sample (not shown here). Thus, I conclude that we can trust the significance of the statistical evidence with regards to variation in expected excess returns at the business cycle frequency. Asymptotically, the presence of integrated regressors do not contaminate the inference on stationary regressors (Watson, 1994). In small samples, highly persistent regressors behave much like integrated regressors.

8.1.2 Splitting the sample

I verify that our results do not arise from a particular time period/observation by splitting the sample in two, using the filtering probability \( f_t \) as the conditioning variable (table 9). The regression coefficients have the same signs across sub-samples, and the level of significance across the two time periods is about the same. However, the results are overall statistically less significant. The interaction term is significant at the 5% level in both sub-periods only in the \( cay \) regressions. Clearly, less data means less power and a simple Chow test cannot reject the hypothesis of equal
regression coefficients across samples, indicating that the lack of significance within each sub-period is indeed due to low statistical power.

8.1.3 Nonlinear, univariate regressions.

Could the interaction effects be due to nonlinearities in the univariate relationship between the forecasting ratios and future excess returns? To investigate this, I regress next quarter excess returns on the forecasting ratios and their (demeaned) value squared.\textsuperscript{15} Table 10 shows there is no evidence of such nonlinearities. Therefore, to replicate the empirical results, we need to be able to separate the notion of bad times as measured by the business cycle and bad times as measured by the forecasting ratios. This indicates that the business cycle represents a risk factor different from the very persistent risk factor established in previous studies. Asset pricing models that aim to account for this evidence therefore need two risk factors that interact nonlinearly and operate at the business cycle and generational frequencies, respectively.

8.1.4 Using Alternative Business Cycle Variables

Are the results a pure GDP phenomenon, or do other business cycle variables yield similar results? Table 11 shows that forecasting regressions using a NBER recession indicator and lagged growth rates in Industrial Production yields similar, significant results. I interpret this as evidence that the findings are indeed due to a business cycle factor, not an idiosyncratic GDP phenomenon.

\textsuperscript{15}I have tried several other specifications such as absolute value, exponential, and cubed in addition the reported squared value regressions. These were also insignificant for all the forecasting ratios.
Table 9
Splitting the Sample

Table 9: Splitting the sample. Quarterly observations. Significance at the 5 percent level is denoted with two asterisks, while significance at the 10 percent level is denoted with 1 asterix. T-statistics are given below the coefficient estimates in parentheses.

\[ r_{t+1} - r_{f,t} = \alpha_1 + \alpha_2 f_t + \beta_1 z_t + \beta_2 f_t z_t + \epsilon_{t+1} \]

Panel A:

<table>
<thead>
<tr>
<th>Sample 1: 52Q1 - 77Q1</th>
<th>Sample 2: 77Q2 - 02Q3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_1 = 101 )</td>
<td>( N_2 = 102 )</td>
</tr>
<tr>
<td>( z_t = c_p t )</td>
<td>( z_t = c_p t )</td>
</tr>
<tr>
<td>( z_t = d_p t )</td>
<td>( z_t = d_p t )</td>
</tr>
<tr>
<td>( z_t = c_{ay} t )</td>
<td>( z_t = c_{ay} t )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Sample 1</th>
<th>Sample 2</th>
<th>T-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_1 )</td>
<td>0.0076 (0.76)</td>
<td>0.0092 (1.26)</td>
<td>(1.76)</td>
</tr>
<tr>
<td>( \alpha_2 )</td>
<td>0.0023 (0.065)</td>
<td>0.017 (0.57)</td>
<td>(0.065)</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>0.047 (1.54)</td>
<td>0.031 (1.56)</td>
<td>(1.54)</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>-0.13 (-1.49)</td>
<td>-0.11 (-1.64)</td>
<td>(-1.49)</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>2.5%</td>
<td>1.4%</td>
<td></td>
</tr>
</tbody>
</table>

Panel B: Chow test of equal coefficients across samples

<table>
<thead>
<tr>
<th>( RSS_{N_1} )</th>
<th>( RSS_{N_2} )</th>
<th>( RSS_{Total} )</th>
<th>( F[4,193]^{14} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( dp )</td>
<td>0.6212</td>
<td>0.6943</td>
<td>1.3435</td>
</tr>
<tr>
<td>( cp )</td>
<td>0.6574</td>
<td>0.6918</td>
<td>1.3547</td>
</tr>
<tr>
<td>( cay )</td>
<td>0.5371</td>
<td>0.6646</td>
<td>1.2177</td>
</tr>
</tbody>
</table>

51
TABLE 10

Univariate forecasting regressions including nonlinear terms

Table 10: This table shows estimates from an OLS regression of excess returns on each of the price ratios. The regressions include a nonlinear term (in this case: squared. Other transformations give qualitatively the same results), in an attempt of picking up any nonlinear relations between these forecasting variables and the equity risk premium. Bold face font indicates significant at the 5 percent level or better. The regression is thus

$r_{t+1} - r_{f,t} = \alpha + \beta_1 z_t + \beta_2 z_t^2 + \epsilon_{t+1}$

<table>
<thead>
<tr>
<th>Price ratio</th>
<th>$\alpha$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$R^2_{adj}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_t = cp_t$</td>
<td>0.012</td>
<td><strong>0.036</strong></td>
<td>0.001</td>
<td>1.3%</td>
</tr>
<tr>
<td></td>
<td>(0.008)</td>
<td>(0.018)</td>
<td>(0.044)</td>
<td></td>
</tr>
<tr>
<td>$z_t = dp_t$</td>
<td>0.010</td>
<td><strong>0.043</strong></td>
<td>0.014</td>
<td>1.9%</td>
</tr>
<tr>
<td></td>
<td>(0.007)</td>
<td>(0.020)</td>
<td>(0.032)</td>
<td></td>
</tr>
<tr>
<td>$z_t = cay_t$</td>
<td>0.012</td>
<td><strong>2.024</strong></td>
<td>-0.904</td>
<td>8.1%</td>
</tr>
<tr>
<td></td>
<td>(0.007)</td>
<td>(0.508)</td>
<td>(19.267)</td>
<td></td>
</tr>
</tbody>
</table>

8.2 Proofs and derivations

8.2.1 Proof of Proposition 1 and 2

Here, I derive the general equilibrium for the economy in section 4, and go on to show that the optimal consumption sharing rule and the stochastic discount factor are the same as when we consider only the problem of optimal allocation of luxury goods. The propositions follow from this.

Aggregate consumption is $Y_t = Y_t^B + PY_t^L$, where $P$ is the relative price of the two goods. Without loss of generality, I assume there is only one agent with utility curvature parameter $\gamma$. This agent then has a minimum consumption level $B_{\gamma} = \int_{i \in \{i: \gamma_i = \gamma\}} B_i \, di$, which is assumed to be external. This agent, indexed $\gamma$, has period utility (we suppress time/state subscripts in this part)

$$u(C_{\gamma}^L, C_{\gamma}^B, X, B_{\gamma}, \gamma) = \frac{1}{1-\gamma} \left( \frac{C_{\gamma}^L}{X} \right)^{1-\gamma} + \frac{\pi}{1-\gamma} \left( \frac{C_{\gamma}^B - B_{\gamma}}{X} \right)^{1-\gamma}$$

over the two goods. The subscript $\gamma$ is short-hand for variables that correspond to investor $\gamma$ and suppresses functional dependency of these variables on relevant state variables. Markets are complete and there are no frictions (although $B$ can be thought of as a proxy for such effects). The intratemporal budget constraint gives

$$\frac{u_{C_{\gamma}^L}}{u_{C_{\gamma}^B}} = \pi P$$

52
Table 11: The table reports estimates from OLS regression of quarterly log excess stock market returns on lagged forecasting variables. The regressors are divided into two groups. First, price ratios, z(t), which are the consumption-price ratio (cp), the dividend-price ratio (dp), and a measure of the consumption-wealth ratio (cay). Second, business cycle variables, h(t), which are semi-annual log growth in Industrial Production, \( IP \), the cyclical component of GDP extracted using the Hodrick-Prescott filter (with \( \lambda = 1600 \)) on data available until time \( t \), \( HP \), and an NBER recession indicator, \( NBER \). The data is from the first quarter of 1952 until the third quarter of 2002. Newey-West corrected standard errors appear below the coefficient estimates. One asterisk means significant at the 10 percent level, while bold font means significant at the 5 percent level or better. All independent variables (z and h) are demeaned. Regression: 

\[
r_{t,t+1} - r_{f,t} = \alpha_1 + \alpha_2 h_t + \beta_1 z_t + \beta_2 h_t z_t + \epsilon_{t+1}
\]

<table>
<thead>
<tr>
<th>Price Ratio</th>
<th>Business Cycle Var.</th>
<th>( \alpha_1 ) (s.e.)</th>
<th>( \alpha_2 ) (s.e.)</th>
<th>( \beta_1 ) (s.e.)</th>
<th>( \beta_2 ) (s.e.)</th>
<th>( R^2_{adj} )</th>
<th>( p - val )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_t = cp_t )</td>
<td>( h_t = \text{none} )</td>
<td>0.012 (0.006)</td>
<td>0.036 (0.018)</td>
<td>1.8%</td>
<td>0.03</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( h_t = IP_{2,t} )</td>
<td></td>
<td>0.011 (0.006)</td>
<td>-0.104 (0.156)</td>
<td>0.030* (0.016)</td>
<td>-1.157 (0.494)</td>
<td>3.8%</td>
<td>0.01</td>
</tr>
<tr>
<td>( h_t = NBER_t )</td>
<td></td>
<td>0.009 (0.006)</td>
<td>0.021 (0.021)</td>
<td>0.035 (0.016)</td>
<td>-0.112 (0.039)</td>
<td>3.8%</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Panel B:

| \( z_t = dp_t \) | \( h_t = \text{none} \) | 0.012 (0.006) | 0.037 (0.018) | 2.3% | 0.02 |
| \( h_t = IP_{2,t} \) | | 0.114 (0.055) | -0.070 (0.167) | 0.030* (0.016) | -1.104 (0.504) | 4.2% | 0.01 |
| \( h_t = NBER_t \) | | 0.009 (0.006) | 0.020 (0.021) | 0.034 (0.016) | -0.107 (0.035) | 4.7% | 0.01 |

Panel C:

| \( z_t = cay_t \) | \( h_t = \text{none} \) | 0.012 (0.006) | 2.029 (0.465) | 8.6% | 0.00 |
| \( h_t = IP_{2,t} \) | | 0.011 (0.007) | -0.055 (0.174) | 2.076 (0.446) | -46.268 (19.610) | 11.3% | 0.00 |
| \( h_t = NBER_t \) | | 0.011 (0.005) | 0.013 (0.017) | 2.129 (0.453) | -4.003 (1.287) | 11.5% | 0.00 |
for all \( \gamma \) and in all states. I.e., \((C^L_\gamma)^{-\gamma} = \pi P (C^B_\gamma - B_\gamma)^{-\gamma}\) for all agents in all states at all times (still suppressing time/state subscripts). For each \( \gamma \), we can express the total consumption as \( C_\gamma = C^B_\gamma + PC^L_\gamma \). These two equations allow us to write \( C^L_\gamma = C^L_\gamma (C_\gamma, B_\gamma; P, \pi) \) and \( C^B_\gamma = C^B_\gamma (C_\gamma, B_\gamma; P, \pi) \). In particular, \[
C^B_\gamma = \frac{1}{1 + P (\pi P)^{-\frac{1}{7}} - \gamma (C^B_\gamma - B_\gamma)}
\]
\[
C^L_\gamma = \frac{(\pi P)^{-\frac{1}{7}}}{1 + P (\pi P)^{-\frac{1}{7}}} (C_\gamma - B_\gamma)
\]

Now, we can write each agent’s utility as a function of total consumption \( C_\gamma \) and the minimum basic good consumption level \( B_\gamma \) only. Since markets are complete, the marginal utility of all agents are proportional in all states of the world. Given a set of social weights \( \lambda \) (which existence is equivalent to assuming an admissible wealth distribution at time \( t = 0 \)), \( \lambda_\gamma \) is the weight the social planner assigns to investor \( \gamma \). The social planner problem is then

\[
\begin{align*}
\sup_{\{C_t(Y_t, S_t, B_t, X_t; \gamma)\}} & E_0 \left[ \int_0^{\infty} e^{-\rho t} \left\{ \int_1^{\infty} \lambda_\gamma u (C_t (Y_t, S_t, B_t, X_t; \gamma)) \, d\gamma \right\} \, dt \right] \\
\text{subject to the resource constraint} & \int_1^{\infty} \frac{C_t (Y_t, S_t, B_t, X_t; \gamma)}{X_t} \, d\gamma \leq \frac{Y_t}{X_t}
\end{align*}
\]

Since \( B_\gamma \)'s are assumed exogenous here and markets are complete (state (and time) dependency is suppressed), the social planner solves the following problem for each state and time

\[
\begin{align*}
\sup_{\{C_\gamma\}} & U_{\lambda} (\{C_\gamma\}) \quad \text{s.t.} \quad \int_1^{\infty} \frac{C_\gamma}{X} \, d\gamma \leq \frac{Y}{X} \\
\downarrow & \\
\inf_{Z \geq 0} & \sup_{\{c_\gamma\}} \int_1^{\infty} \lambda_\gamma u (c_\gamma; B_\gamma, \gamma) \, d\gamma - Z e^{\omega} \left[ \int_1^{\infty} c_\gamma \, d\gamma - 1 \right]
\end{align*}
\]

where \( c_\gamma = \frac{C_\gamma}{Y} \) and \( \omega = \ln W \equiv \ln \frac{Y}{X} \). Note \( \frac{\partial C^L_\gamma (C_\gamma, B_\gamma; P, \pi)}{\partial c_\gamma} = \frac{\partial C^L_\gamma (C_\gamma, B_\gamma; P, \pi)}{\partial C_\gamma} Y \). The first order condition for each agent \( \gamma \) in this problem is

\[
\lambda_\gamma \left( k_L Y X^{1-\gamma} (C^L_\gamma)^{-\gamma} + k_B Y X^{1-\gamma} (C^B_\gamma - B_\gamma)^{-\gamma} \right) = Z e^{\omega}
\]
where \( k_L = \frac{(\pi P)^\frac{1}{\gamma}}{1 + P(\pi P)^{-\frac{1}{\gamma}}} \) and \( k_B = \frac{1}{1 + P(\pi P)^{-\frac{1}{\gamma}}} \). Since \((C_L^\gamma)^{-\gamma} = \pi P (C_B^\gamma - B^\gamma)^{-\gamma}\), we can write

\[
\lambda^\gamma \left( k_L + \frac{1}{\pi P} k_B \right) (C_L^\gamma)^{-\gamma} = Z e^{\omega} e^{-\omega} X^{-\gamma}
\]

Define \( f(\gamma) = \lambda^\gamma (k_L + \frac{1}{\pi P} k_B) \), \( c_L^\gamma = \frac{C_L^\gamma}{1+P} \) and \( \tilde{S} \equiv \frac{Y_L}{\gamma} \). Then

\[
f(\gamma) (c_L^\gamma)^{-\gamma} = Z (Y_L)^\gamma X^{-\gamma} = Z \tilde{S} Y^\gamma X^{-\gamma} = Z \tilde{S} W^\gamma
\]

which yields

\[
c_L^{\gamma,*} = f(\gamma)^\frac{1}{\gamma} e^{-\frac{1}{\gamma} \tilde{S} - \omega} = f(\gamma)^\frac{1}{\gamma} e^{-\frac{1}{\gamma} \tilde{S} - \omega L}
\]

Where I have defined \( \omega^L \equiv \tilde{S} + \omega \) as a "new" composite state variable. The resource constraint over luxury goods is

\[
\int_1^\infty c_L^{\gamma,*} d\gamma = 1
\]

\[
\int_1^\infty f(\gamma)^\frac{1}{\gamma} e^{-\frac{1}{\gamma} \tilde{S} - \omega} d\gamma = e^{\tilde{S} + \omega} = e^{\omega^L}
\]

(31)

Since utility is separable across goods and markets are frictionless, the marginal utility over luxury good consumption of an arbitrary agent is sufficient for asset pricing purposes. However, we need the resource constraint over basic goods to also be satisfied. Optimal consumption of basic goods for agent \( \gamma \) is

\[
C_B^{\gamma,*} = (\pi P)^\frac{1}{\gamma} C_L^{\gamma,*} + B^\gamma
\]

(32)

The resource constraint over this good is

\[
\int_1^\infty C_B^{\gamma,*} d\gamma = Y^B
\]

\[
\int_1^\infty (\pi P)^\frac{1}{\gamma} C_L^{\gamma,*} + B^\gamma d\gamma = Y^B
\]

\[
B_A = Y^B - \int_1^\infty (\pi P)^\frac{1}{\gamma} C_L^{\gamma,*} d\gamma
\]

(33)

where \( B_A = \int_1^\infty B^\gamma d\gamma \) is the aggregate level of required basic good consumption. Solving for agents'
optimal consumption, and thus obtaining the pricing kernel, requires solving for \( \tilde{s} = \ln \left( \frac{Y}{\bar{Y}} \right) \). If \( \pi P = 1 \), as assumed in section 4, \( \bar{B}_A = Y^B - Y^L \). Since \( S \equiv \frac{Y - \bar{B}_A}{Y} \) is given exogenously, \( P \) uniquely determines \( Y^L \) and \( Y^B \) in equilibrium. In particular, \( S = \frac{Y - \bar{B}_A}{Y} = \frac{Y - Y^B + Y^L}{Y} \). Since \( Y = Y^B + PY^L \),
\[
S = \frac{(1 + P) Y^L}{Y} = (1 + P) \tilde{S}
\]
where \( \tilde{S} = \frac{Y^L}{\bar{Y}} \) as defined above. Thus, \( ds_t = d\tilde{s}_t \), which means we have analytic expressions for the equilibrium, dynamic behavior of both state variables. I will in the following not distinguish between \( s \) and \( \tilde{s} \) since they are equal up to a constant.

With the resource constraints satisfied for both goods, the total resource constraint is satisfied. The Inada conditions hold, so this is an interior solution when the first order conditions are satisfied. Thus, we have a solution to the social planner problem given the weights \( \lambda \gamma \). Notice that individual optimal consumption of luxury goods is not a function of the individual \( B_\gamma \)'s but only of the aggregate \( B_A \) through \( s_t \). This is due to the assumption that \( B_\gamma \) is external. Thus, specifying \( B_\gamma \) across agents do not have pricing implications, but does have implications for the distribution of wealth and the dynamics of individual optimal total consumption.

**The Stochastic Discount Factor** The stochastic discount factor is proportional to the social planner’s marginal utility over aggregate consumption. From Duffie (1996) we know this is proportional to the Lagrange multiplier over the resource constraint (28). In our case this is \( Z_t X_t \), where \( Z_t \) is defined in (31). Therefore,
\[
M_t \propto e^{-\rho t + z_t (\omega^L_t) + \omega_t - y_t}
\]
where \( \omega^L_t = s_t + \omega_t \).

The optimal allocation of consumption and the stochastic discount factor we obtain from the complete problem are the same as if we just consider the partial problem of optimal luxury good consumption sharing. This is because the relative price \( P \) is constant, so the partial marginal utilities are always proportional since the commitment levels are exogenous and assumed to follow processes consistent with a constant \( P \). It is then immediate that the partial problem over optimal allocation over luxury goods only yields the same result.

**Proof of Proposition 3** The stochastic discount factor is
\[
M_t \propto \exp \left( -\rho t + z \left( \omega^L_t \right) + \omega_t - y_t \right)
\]
It is well known (see e.g. Duffie 1996) that the price of risk process \( \Lambda_t \) and the risk free rate is given by

\[
\frac{dM_t}{M_t} = -r_{f,t} dt - \Lambda_t dW_t
\]

From an application of Ito’s Lemma on the \( z_t \) variable, we have

\[
dz_t(s_t, \omega_t) = \left\{ \frac{\partial z}{\partial \omega} L_t \gamma_{s,t} + \frac{1}{2} \frac{\partial^2 z}{\partial \omega^2} \left( \sigma_{\omega t} + \sigma_{s t} \right)^2 \right\} dt + \frac{\partial z}{\partial \omega} L_t \left( \sigma_{\omega t} + \sigma_{s t} \right) dB_t
\]

The risk free rate is given by an application of Ito’s Lemma on the sdf.

\[
r_{t,f} = \rho - \mu z_t - \mu_{\omega t} + \mu - \frac{1}{2} \left( \frac{\partial z}{\partial \omega} \right)^2 \left( \sigma_{\omega t} + \sigma_{s t} \right)^2
\]

\[
= \rho - \frac{\partial z}{\partial \omega} \mu_{s,t} - \frac{\partial z}{\partial \omega} \mu_{\omega,t} - \frac{1}{2} \frac{\partial^2 z}{\partial (\omega^2)} \left( \sigma_{\omega t} + \sigma_{s t} \right)^2 - \mu_{\omega t} + \mu - \frac{1}{2} \left( \frac{\partial z}{\partial \omega} \right)^2 \left( \sigma_{\omega t} + \sigma_{s t} \right)^2
\]

\[
= \rho + \mu - \frac{\partial z}{\partial \omega} \mu_{s,t} - \left( \frac{\partial z}{\partial \omega} + 1 \right) \mu_{\omega t} - \frac{1}{2} \left( \frac{\partial^2 z}{\partial \omega^2} \right)^2 + \left( \frac{\partial z}{\partial \omega} \right)^2 \left( \sigma_{\omega t} + \sigma_{s t} \right)^2
\]

Define

\[
\gamma_{A,t} = -\frac{\partial z}{\partial \omega} \mu_{s,t} - \left( \frac{\partial z}{\partial \omega} + 1 \right) \mu_{\omega t} - \frac{1}{2} \left( \frac{\partial^2 z}{\partial \omega^2} \right)^2 + \left( \frac{\partial z}{\partial \omega} \right)^2 \left( \sigma_{\omega t} + \sigma_{s t} \right)^2
\]

Finally, the price of risk equals the conditional Sharpe ratio

\[
\Lambda_t = -\frac{\partial z}{\partial \omega} \left( \sigma_{\omega t} + \sigma_{s t} \right) = \gamma_{A,t} \sigma \left( 1 + \lambda \left( s_t \right) \right)
\]

**Proof of Proposition 4** In the Cochrane and Campbell economy, the representative agent has preferences

\[
U(Y) = E_t \int_t^\infty \frac{1}{1 - \gamma} \left( Y_s - X_s \right)^{1-\gamma} ds
\]

Where \( Y \) is aggregate consumption and \( X \) is an external difference habit. Define the surplus
consumption ratio as \( S = \frac{Y - X}{Y} = \frac{Y^L}{Y} \), where \( Y^L = Y - X \). Note, the surplus consumption ratio is defined the same as our luxury share process. This is the same economy as our economy, but with no ratio habit and only one agent with risk aversion \( \gamma \). Therefore the fraction of surplus consumption out of total surplus consumption consumed by this agent is 1; \( h_{\gamma,t} = \frac{1}{\gamma} \ln \left( f(\gamma) \right) - \frac{1}{\gamma} z(\omega_t^L) - \omega_t - s_t = 1 \) and \( \gamma_{A,t} = \gamma \). This yields the same expression for the Sharpe ratio for any \( \gamma \) since the relative consumption process does not affect the conditional price of risk. However, it does affect the risk free rate, except for the case \( \gamma = 1 \), which means the representative agent ignores the standard of living process.

In the Chan and Kogan economy, there are a continuum of agents with \( \gamma \in [1, \infty) \), but only a single good in the economy. In terms of the economy presented here, \( Y^L = Y \) and \( s_t = 0 \), always. In this case, the economy presented here is isomorphic to the Chan and Kogan economy.

8.3 Example economy: The case of two agents: \( \gamma_2 = 2\gamma_1 \)

From the social planner’s FOC over luxury good consumption, we have for each state and time

\[
b^{\gamma_1} X^{\gamma_1-1} \left( C_1^L \right)^{-\gamma_1} = X^{2\gamma_1-1} \left( C_2^L \right)^{-2\gamma_1}
\]

where \( f(\gamma_2) = 1 \) without loss of generality and \( b = f(\gamma_1)^{\frac{1}{\gamma_1}} \). Let \( \gamma_1 \equiv \gamma \). Then

\[
b^{\gamma} X^{\gamma-1} \left( C_1^L \right)^{-\gamma} = X^{2\gamma-1} \left( C_2^L \right)^{-2\gamma}
\]

\[
bX^{-1} \left( C_2^L \right)^2 + C_2^L - bY^L = 0
\]

Solving this yields

\[
C_2^{L*,t} = \frac{X}{2b} \left( \sqrt{1 + 4bX^{-1}Y^L} - 1 \right)
\]

Remember that \( e^{\gamma + \omega_t} = \frac{Y_t^L}{X_t} \). Now,

\[
C_{2,t}^{L*,t} = \frac{Y_t^L}{2b e^{\gamma + s_t}} \left( \sqrt{1 + 4b e^{\gamma + s_t}} - 1 \right)
\]

\[
C_{1,t}^{L*,t} = Y_t^L - C_{2,t}^{L*,t}
\]

Define \( H_t(\omega_t, s_t) \equiv \frac{C_{1,t}^{L*,t}}{Y_t^L} \) as the equilibrium luxury good consumption of agent 1 as a fraction of aggregate luxury good consumption. Now, plugging this into agent 1’s marginal utility over luxury
goods yields

\[ MC_{1,t} = e^{-\rho t} X^{\gamma - 1} \left( Y_t^{-1} \right)^{-\gamma} \left( \frac{C_{1,t}^{L_*}}{Y_t^{L_*}} \right)^{-\gamma} = e^{-\rho t - (\gamma - 1)\omega_t - y_t - \gamma s_t - \gamma h_t} \]

The stochastic discount factor is proportional to the marginal utility above.