Long-Run Risk through Consumption Smoothing

Georg Kaltenbrunner* and Lars Lochstoer†

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Abstract

Whenever agents have access to a production technology they will engineer optimal consumption paths. This is usually perceived as making the task of explaining asset prices much harder. We show that this is not the case in a standard production economy model where consumers have Epstein-Zin preferences and dislike negative shocks to future economic growth prospects. Endogenous consumption smoothing increases the price of risk in this economy as it induces highly persistent time-variation in expected aggregate consumption growth (long-run risk), even though technology follows a random walk. The asset pricing properties of the production economy model with i.i.d. shocks to technology are therefore actually better than the asset pricing properties of the exchange economy model counterpart with i.i.d. shocks to consumption. The model identifies an observable proxy for otherwise hard to measure expected consumption growth. Using this proxy, we test and find support for key predictions of our model in the time-series of consumption growth and the cross-section of stock returns.

*London Business School. Mailing address: IFA, 6 Sussex Place Regent’s Park, London, United Kingdom NW1 4SA. Email: gkaltenbrunner.phd2003@london.edu
†Corresponding author. London Business School. Mailing address: IFA, 6 Sussex Place Regent’s Park, London, United Kingdom NW1 4SA. Email: llochstoer@london.edu
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1 Introduction

Asset prices and macroeconomic aggregates such as consumption and investment are intrinsically linked because financial markets provide the mechanism with which we allocate savings to investments. However, many asset pricing models specify exchange economies (Lucas, 1978), where the process for aggregate consumption is exogenous and investment does not play an explicit role. One important reason for this modeling choice is that access to a production technology enables agents to engineer optimal consumption paths. This is usually perceived as making the task of explaining asset prices much harder. Rouwenhorst (1995) nicely summarizes this point: "[... ] it is more difficult to explain substantial risk premiums in a production economy, because consumption choices are endogenously determined and become smoother as risk aversion increases."

We show that this general intuition does not hold in a standard production economy model where consumers have Epstein-Zin preferences and dislike negative shocks to future economic growth prospects. Unlike the case of power utility preferences, where risk is only associated with the shock to realized consumption growth, investors in this economy also dislike negative shocks to expected consumption growth and consequentially demand a premium for holding assets correlated with this shock. The latter source of risk has been labelled "long-run risk" in previous literature (Bansal and Yaron, 2004). When the log technology process follows a random walk, endogenous consumption smoothing increases the price of risk in the production economy model exactly because it increases the amount of long-run risk in the economy. That is, consumption smoothing induces highly persistent time-variation in expected consumption growth rates. The asset pricing properties of the production economy model are then actually better than the asset pricing properties of the exchange economy model counterpart with i.i.d. shocks to consumption.

Why does the consumer optimally choose a consumption process that leads to a high price of risk? The price of risk is related to risk across states, while the agent maximizes the level of expected utility which also is a function of substitution across time. The agent thus trades off the benefit of shifting consumption across time with the cost of higher volatility.

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1 For an extensive discussion of this point, refer to Rouwenhorst (1995), Lettau and Uhlig (2000), Uhlig (2004), and Cochrane (2005), amongst others.

2 Epstein-Zin preferences provide a convenient separation of the elasticity of intertemporal substitution ($\psi$) from the coefficient of relative risk aversion ($\gamma$), which are forced to $\gamma = \frac{1}{\psi}$ in the power utility case. If $\gamma > \frac{1}{\psi}$, investors prefer early resolution of uncertainty and are averse to time-varying expected consumption growth.
of marginal utility across states. Asset prices in the production economy simply reflect the optimal outcome of this trade-off. A higher elasticity of intertemporal substitution results in more substitution across time at the expense of additional risk across states, and thus a higher price of risk, higher Sharpe ratios, and a lower and less volatile risk-free rate.

In equilibrium, time-varying expected consumption growth turns out to be a small, but highly persistent fraction of realized consumption growth. When the model is calibrated to fit standard macroeconomic moments, the endogenous expected consumption growth rate process is quantitatively very close to the exogenous processes that have been specified in the recent asset pricing literature (see, e.g., Bansal and Yaron, 2004). Note that this result is of particular interest since it is very difficult to empirically distinguish a small predictable component of consumption growth from i.i.d. consumption growth given the short sample of data we have available (see Harvey and Shepard, 1990, and Hansen, Heaton and Li, 2005, amongst others). Bansal and Yaron (2004), for instance, calibrate a process for consumption growth with a highly persistent trend component and demonstrate that their process can match a number of moments of aggregate consumption growth. In lieu of robust empirical evidence on this matter, the model presented in this paper provides a theoretical justification for the previously proposed long-run risk dynamics of aggregate consumption growth based on a standard production economy setup. We conclude that simple consumption smoothing in an economy with i.i.d. technology growth naturally induces long-run consumption risk. Long-run consumption risk is therefore not an esoteric assumption for aggregate consumption dynamics. On the contrary, it is the natural assumption, given our standard theoretical models, for exogenous consumption growth processes in exchange economy models.

The persistence of the technology shocks is crucial for the asset pricing implications of long-run risk in the economy. In short, permanent shocks lead to time-varying expected consumption growth which increases the price of risk in the economy, while transitory shocks lead to time-varying expected consumption growth that decreases the price of risk. The intuition for this is as follows. A permanent positive shock to productivity implies a permanently higher optimal level of capital. As a result, investors increase investment in order to build up a higher capital stock. High investment today implies low current consumption, but high future consumption. Thus, expected consumption growth is high. The higher investors’ elasticity of intertemporal substitution, the more willing investors are to substitute consumption today for higher consumption in the future, and the stronger this effect is. Since agents in this economy dislike negative shocks to future economic growth prospects, both shocks to expected consumption growth and realized consumption growth are risk factors.
Furthermore, the shocks are positively correlated and thus reinforce each other. In this case, endogenous consumption smoothing increases the price of risk in the economy.

If, on the other hand, shocks to technology are transitory, the endogenous long-run risk in general decreases the price of risk in the economy. A transitory, positive shock to technology implies that technology is expected to revert back to its long-run trend. Thus, if realized consumption growth is high, expected future long-run consumption growth is low as consumption also reverts to the long-run trend. The shock to expected future consumption growth is in this case negatively correlated with the shock to realized consumption growth. The long-run risk component acts as a hedge for realized consumption risk and therefore decreases the price of risk.\(^3\) We evaluate the quantitative effects of transitory vs. permanent technology shocks on aggregate macroeconomic and financial moments with calibrated versions of our model and show that the model can match the high historical Sharpe ratio of the aggregate stock market and the level of the risk free rate with a low coefficient of relative risk aversion if technology shocks are permanent and the elasticity of intertemporal substitution is relatively high.

But what if the elasticity of intertemporal substitution is so low that investors actually like shocks to expected consumption growth? Then we would expect the model with transitory technology shocks, where shocks to realized consumption growth are negatively correlated with shocks to long-run expected consumption growth, to also generate a high price of risk. We show that this is indeed the case. A high elasticity of substitution is therefore not necessary to generate endogenous long-run risk. Unfortunately, very low levels of the elasticity of intertemporal substitution imply a much too high level of the risk-free rate (Weil, 1989). For this reason, we mainly focus on the case of higher elasticity of intertemporal substitution.

The production economy model relates the aggregate level of technology (total factor productivity), consumption, and investment to the dynamic behavior of aggregate consumption growth. We use this link to derive new testable implications. Our model implies that the ratio of total factor productivity to consumption is a good proxy for the otherwise hard to measure expected consumption growth rate. We find empirical support for this by showing

\(^3\)This description is intentionally loose to emphasize the intuition. The consumption response to transitory technology shocks is actually hump-shaped. Thus, a positive shock to realized consumption growth is followed by high expected consumption growth in the near term, but lower expected consumption growth in the long term - the negative correlation arises at lower frequencies. The low frequency effect dominates for standard values of the time-discounting parameter and leads to a lower price of risk unless the transitory shocks are extremely persistent.
that the ratio of log total factor productivity to consumption forecasts future consumption growth over long horizons. We furthermore test a linear approximation of the model on the cross-section of stock returns and show, using the above proxy, that shocks to expected consumption growth are a priced risk factor that substantially improves the ability of the Consumption CAPM to explain the cross-section of stock returns.

We proceed as follows. We start by providing an overview of related literature. Then we develop and interpret our model. In section 4 we calibrate and solve the model, demonstrate and interpret results, and provide intuition. In section 5 we test some empirical implications of our model. Section 6 concludes.

2 Related Literature

This paper is mainly related to three strands of the literature: the literature on consumption smoothing, the literature on long-run risk, and the literature that aims to jointly explain macroeconomic aggregates and asset prices.

It is well-known that (risk averse) agents, if they can, smooth consumption over time. The permanent income hypothesis of Friedman (1957) is the classic reference. Hall (1978) is a seminal empirical investigation of this hypothesis. Hall shows that consumption should approximately follow a random walk and finds support for this in the data. The results in our paper are consistent with Hall: We also find that consumption should be very close to a random walk. But, different from Hall, we emphasize that consumption growth has a small, highly persistent, time-varying component. Time-variation in expected growth rates, arising from consumption smoothing in production economy models, has been pointed out before. For example, Den Haan (1995) demonstrates that the risk-free rate in production economy models is highly persistent (close to a random walk) even when the level of technology is i.i.d.

Bansal and Yaron (2004) show that a small, persistent component of consumption growth can have quantitatively important implications for asset prices if the representative agent has Epstein-Zin (1989) preferences. Bansal and Yaron term this source of risk "long-run risk" and show that it can explain many aspects of asset prices. They specify exogenous processes for dividends and consumption with a slow-moving expected growth rate component and demonstrate that the ensuing long-run consumption risk greatly improves their
model’s performance with respect to asset prices without having to rely on, e.g., habit formation and the high relative risk aversion such preferences imply. We show that the process for consumption Bansal and Yaron assume as exogenous can be generated endogenously in a standard production economy model with Epstein-Zin preferences and the same preference parameters Bansal and Yaron use. Since it is very difficult to empirically distinguish between i.i.d. consumption growth and consumption growth with a very small, highly persistent time-varying component, this result is of particular importance for the Bansal and Yaron framework. Hansen, Heaton and Li (2005) emphasize this point in their study of the impact of long-run risk on the cross-section of stock returns. We also consider the implications for aggregate investment, which Bansal and Yaron abstract from, and we endogenize the aggregate dividend process.

A recent paper that generates interesting consumption dynamics is due to Panageas and Yu (2006). These authors focus on the impact of major technological innovations and real options on consumption and the cross-section of asset prices. They assume, as do we, the technology process to be i.i.d. The major technological innovations, however, are assumed to occur at a very low frequency (about 20 years), and are shown to carry over into a small, highly persistent component of aggregate consumption. In that sense, Panageas and Yu assume, contrary to us, the frequency of the predictable component of consumption growth. Moreover, time-variation in expected consumption growth (long-run risk) is not itself a priced risk factor in the Panageas and Yu model because the representative agent does not have Epstein-Zin (1989) preferences, but external ratio-habit as in Abel (1990). Panageas and Yu require that investment is irreversible, whereas we allow for a convex adjustment cost function. Also, since investment in their model means paying a "gardener" to plant a tree, their model does not have a clear separation of investment and labor income. Parker and Julliard (2005) find that the CCAPM works well when consumption growth is measured over longer horizons. This is consistent both with frictions to consumption adjustment and the presence of long-run risks.

There are quite a few papers before Bansal and Yaron (2004) that emphasize a small, highly persistent component in the pricing kernel. An early example is Backus and Zin (1994) who use the yield curve to reverse-engineer the stochastic discount factor and find that it has high conditional volatility and a persistent, time-varying conditional mean with very low volatility. These dynamics are also highlighted in Cochrane and Hansen (1992). This is exactly the dynamic behavior generated endogenously by the models considered in this paper, and as such the paper complements the above earlier studies. The use of
Epstein-Zin (1989) preferences provides a justification for why the small, slow-moving time-variation in expected consumption growth generates high volatility of the stochastic discount factor. These preferences have become increasingly popular in the asset pricing literature. By providing a convenient separation between the coefficient of relative risk aversion and the elasticity of intertemporal substitution, they help to jointly explain asset market data and aggregate consumption dynamics. An early implementation is Epstein and Zin (1991), while Malloy, Moskowitz and Vissing-Jorgensen (2005) and Yogo (2006) are more recent, successful examples.

Our paper is also part of the strand of the asset pricing literature that tries to jointly explain asset prices and aggregate consumption. The first models in the fold are due to Jermann (1998) and Boldrin, Christiano, Fisher (1999, 2001). Both models rely on two complementary features that enable them to match basic asset pricing moments: (i) households have to be sufficiently sensitive to consumption risk - both Jermann as well as Boldrin, Christiano, Fisher use habit preferences, and (ii) households have to be prevented from using their investment decision to rid themselves of most of the consumption risk they might otherwise face - Jermann imposes capital adjustment costs on the economy, while Boldrin, Christiano, Fisher propose a two-sector economy and assume that capital can not be reallocated across sectors in response to a technology shock. Both Jermann and Boldrin, Christiano, Fisher manage to match with their models most of the basic asset pricing moments, such as the equity premium and the equity return volatility, as well as basic moments of macroeconomic time series. However, the models suffer from the usual drawbacks of habit preferences: A much too volatile risk-free rate, and (implicitly) very high levels of relative risk aversion. The model we propose is better in the sense that we can match Sharpe ratios without the risk-free rate being counterfactually volatile and without excessive assumptions on preference parameters. On the other hand, in our model the level of the equity premium turns out to be too low because equity returns are not volatile enough.

Tallarini (2000) proposes a model that is closely related to our setup. In essence, Tallarini restricts himself to a special case of our model with the elasticity of intertemporal substitution fixed at unity and no capital adjustment costs. By increasing the coefficient of relative risk aversion to very high levels Tallarini manages to match some asset pricing moments such as the market price of risk (Sharpe ratio) as well as the level of the risk-free rate, while equity premium and return volatilities in his model remain basically zero. We differ from Tallarini in that our focus is on changing the elasticity of intertemporal substitution and the
implications for the pricing and existence of long-run risk. Relative to the Tallarini setup we show that (moderate) capital adjustment costs together with an elasticity of intertemporal substitution greater than unity can dramatically improve the model’s ability to match asset pricing moments. We confirm Tallarini’s conclusion that the behavior of macroeconomic time series is driven by the elasticity of intertemporal substitution and largely unaffected by the coefficient of relative risk aversion. However, we do not confirm a "separation theorem" of quantity and price dynamics (Cochrane, 2005, p. 46). On the contrary, as we change the elasticity of substitution in our model, both macroeconomic quantity and asset price dynamics are greatly affected.

3 The Model

The model is a standard real business cycle model (Kydland and Prescott, 1982, and Long and Plosser, 1983). There is a representative firm with Cobb-Douglas production technology and capital adjustment costs, and a representative agent with Epstein-Zin (1989) preferences. Our objective is to demonstrate that standard production economy models endogenously give rise to long-run consumption risk and that this long-run risk can improve the performance of these models in replicating important moments of asset prices. To that end we keep both production technology as well as the process for total factor productivity as simple and as standard as possible. In particular, we do not assume any propagation mechanisms such as time-to-build or labor market frictions. We describe the key components of our model in turn.

The Representative Agent. We assume a representative household whose preferences are in the recursive utility class of Epstein and Zin (1989):

$$U_t(C_t) = \left\{ (1 - \beta) C_t^{\frac{1-\gamma}{\theta}} + \beta (E_t[U_{t+1}^{1-\gamma}])^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}},$$

where $E_t$ denotes the expectation operator, $C_t$ denotes aggregate consumption, $\beta$ the discount factor, and $\theta = \frac{1-\gamma}{1-1/\psi}$. Epstein and Zin show that $\gamma$ governs the coefficient of relative risk aversion and $\psi$ the elasticity of intertemporal substitution. These preferences thus have the useful property that it is possible to separate the agent’s relative risk aversion from the elasticity of intertemporal substitution, unlike the standard power utility case where $\gamma = \frac{1}{\psi}$. If $\gamma \neq \frac{1}{\psi}$, these preferences are no longer time-additive and agents care about the temporal
distribution of risk - a feature that is central to our analysis. We focus on the case where $\gamma > \frac{1}{\psi}$. In this case investors have a preference for early resolution of uncertainty. As a result, investors dislike fluctuations in future economic growth prospects (i.e., fluctuations in expected consumption growth). We discuss this property and its implications in more detail below.

**The Stochastic Discount Factor and Risk.** The stochastic discount factor, $M_{t+1}$, is the ratio of the representative agent’s marginal utility between today and tomorrow: $M_{t+1} = \frac{U'(C_{t+1})}{U'(C_t)}$. Using a recursive argument, Epstein and Zin (1989) show that:

$$\ln M_{t+1} \equiv m_{t+1} = \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1 - \theta) r_{a,t+1},$$

(2)

where $\Delta c_{t+1} = \ln \frac{C_{t+1}}{C_t}$ and $r_{a,t+1} \equiv \ln \frac{A_{t+1} + C_{t+1}}{A_t}$ is the return on the total wealth portfolio with $A_t$ denoting total wealth at time $t$. If $\gamma = \frac{1}{\psi}$, $\theta = \frac{1-\gamma}{1-1/\psi} = 1$, and the stochastic discount factor collapses to the familiar power utility case, where shocks to realized consumption growth are the only source of risk in the economy. However, if $\gamma \neq \frac{1}{\psi}$, the return on the wealth portfolio appears as a risk factor. Persistent time-variation in expected consumption growth (the expected "dividends" on the total wealth portfolio) induces higher volatility of asset returns (Barsky and DeLong, 1993). Thus, the return on any asset is a function of the dynamic behavior of realized and expected consumption growth (Bansal and Yaron, 2004).

Depending on the sign of $\theta$ and the covariance between realized consumption growth and the return on the total wealth portfolio, the volatility of the stochastic discount factor (i.e., the price of risk in the economy) can be higher or lower relative to the benchmark power utility case. We show later how this covariance, and thus the amount of long-run risk due to endogenous consumption smoothing, changes with the persistence of the technology shock.

We focus on the case where investors prefer early resolution of uncertainty ($\gamma > \frac{1}{\psi}$) and therefore dislike fluctuations in future economic growth prospects. In the appendix, we explain in more detail how a preference for early resolution of uncertainty translates into aversion of time-varying expected consumption growth. We will refer to the volatility of expected future consumption growth rates as "long-run risk".

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*Note that our representative household’s total wealth portfolio is composed of the present value of future labor income in addition to the value of the firm.*
**Technology.** There is a representative firm with a Cobb-Douglas production technology:

\[ Y_t = (Z_t H_t)^{1-\alpha} K_t^\alpha, \]  

where \( Y_t \) denotes output, \( K_t \) the firm’s capital stock, \( H_t \) the number of hours worked, and \( Z_t \) denotes the (stochastic) level of aggregate technology. This constant returns to scale and decreasing marginal returns production technology is standard in the macroeconomic literature. We assume households to supply a constant amount of hours worked (following, e.g., Jermann, 1998) and normalize \( H_t = 1. \) \(^5\) The productivity of capital and labor depends on the level of technology, \( Z_t \), which is the exogenous driver of the economy. We model log technology, \( z \equiv \ln (Z) \), both as a random walk with drift, and as an AR(1) with a time trend:

\[
\begin{align*}
    z_{t+1} &= \mu + z_t + \sigma_z \varepsilon_{t+1}, \\
    \varepsilon_t &\sim N(0,1),
\end{align*}
\]

or:

\[
\begin{align*}
    z_{t+1} &= \mu t + \varphi z_t + \sigma_z \varepsilon_{t+1}, \\
    \varepsilon_t &\sim N(0,1), \ |\varphi| < 1.
\end{align*}
\]

Thus, (4) implies that technology shocks are permanent whereas (5) implies that technology shocks are transitory. Both specifications are common in the literature. We discuss the two specifications separately.

**Capital Accumulation and Adjustment Costs.** The agent can shift consumption from today to tomorrow by investing in capital. The firm accumulates capital according to the following law of motion:

\[ K_{t+1} = (1-\delta) K_t + \phi \left( \frac{I_t}{K_t} \right) K_t, \]

\(^5\)Let \( W_t \) denote wages. Since \( W_t H_t = (1-\alpha) Y_t \) in equilibrium, a constant labor supply ensures procyclical labor productivity and thus pro-cyclical wages, consistent with the data. Assuming households to supply a constant amount of labor amounts to assuming that households incur no dis-utility from working longer hours.
where \( I_t \) is aggregate investment and \( \phi(\cdot) \) is a positive, concave function, capturing the notion that adjusting the capital stock rapidly by a large amount is more costly than adjusting it step by step. We follow Jermann (1998) and Boldrin, Christiano, Fisher (1999) and specify:

\[
\phi \left( \frac{I_t}{K_t} \right) = \frac{\alpha_1}{1 - 1/\xi} \left( \frac{I_t}{K_t} \right)^{(1-1/\xi)} + \alpha_2, \tag{7}
\]

where \( \alpha_1, \alpha_2 \) are constants and \( \alpha_1 > 0 \). The parameter \( \xi \) is the elasticity of the investment-capital ratio with respect to Tobin’s \( q \). If \( \xi = \infty \) the capital accumulation equation reduces to the standard growth model accumulation equation without capital adjustment costs.

Each period the firm’s output, \( Y_t \), can be used for either consumption or investment. Investment increases the firm’s capital stock, which in turn increases future output. High investment, however, means the agent must forego some consumption today. Let \( C_t, D_t, \) and \( W_t \) denote aggregate consumption, dividends, and wages, respectively. The relations \( C_t = Y_t - I_t \) and \( D_t = \alpha Y_t - I_t \) illustrate how the dynamic behavior of consumption and dividends is determined by the firm’s investment strategy, or equivalently the agent’s optimal savings decision.\(^7\)

**The Return to Investment and the Firm’s Problem.** Let \( \Pi (K_t, Z_t; W_t) \) be the operating profit function of the firm, where \( W_t \) are equilibrium wages. Firm dividends equal operating profits minus investment:

\[
D_t = \Pi (K_t, Z_t; W_t) - I_t. \tag{8}
\]

The firm maximizes firm value. Let \( M_{t,t+1} \) denote the stochastic discount factor. The firm’s problem is then:

\[
\max_{\{I_t, K_{t+1}, W_t\}_{t=0}^T} E_0 \sum_{t=0}^T M_{0,t} D_t, \tag{9}
\]

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\(^6\)In particular, we set \( \alpha_1 = (\exp(\mu) - 1 + \delta)^{1/\xi} \) and \( \alpha_2 = \frac{1}{\xi-1} (1 - \delta - \exp(\mu)) \). It is straightforward to verify that \( \phi \left( \frac{I_t}{K_t} \right) > 0 \) and \( \phi'' \left( \frac{I_t}{K_t} \right) < 0 \) for \( \xi > 0 \) and \( \frac{I_t}{K_t} > 0 \). Furthermore, \( \phi \left( \frac{I_t}{K_t} \right) = \frac{I_t}{K_t} \) and \( \phi' \left( \frac{I_t}{K_t} \right) = 1 \), where \( \frac{I_t}{K_t} \) is the steady state investment-capital ratio.

\(^7\)\( C_t = Y_t - I_t \) states that output must be used for either consumption or investment. Wages are given by the marginal productivity of labor: \( W_t = (1 - \alpha) Y_t \). Since \( C_t = D_t + W_t \), we have that \( D_t = \alpha Y_t - I_t \).
where $E_t$ denotes the expectation operator conditioning on information available up to time $t$. In the appendix, we demonstrate that the return on investment can be written as:

$$R_{t+1}^I = \phi' \left( \frac{I_t}{K_t} \right) \left( \Pi_K (K_{t+1}, Z_{t+1}; W_{t+1}) + \frac{1 - \delta + \phi \left( \frac{I_{t+1}}{K_{t+1}} \right)}{\phi' \left( \frac{I_{t+1}}{K_{t+1}} \right)} - \frac{I_{t+1}}{K_{t+1}} \right). \quad (10)$$

This return to the firm’s investment is equivalent to the firm’s equity return in equilibrium, $R_{t+1}^E \equiv \frac{D_{t+1} + P_{t+1}}{P_t}$, where $P_t$ denotes the net present value of a claim on all future dividends (see, e.g., Zhang, 2005 and Kaltenbrunner, 2006).

### 3.1 The Endogenous Consumption Choice

As can be seen from the stochastic discount factor (see eq. (2)), there are two sources of risk in this economy. The first is the shock to realized consumption growth, which is the usual risk factor in the Consumption CAPM. The second risk factor is the shock to the return on total wealth. Total wealth is the sum of human and financial capital, and the dividend to total wealth is consumption. Assume for the moment that future expected consumption growth and returns are constant. Total wealth, $A_t$, is then given by

$$A_t = \frac{C_t}{r_a - g_c} \quad (11)$$

where $r_a$ is the expected return to wealth and $g_c$ is expected consumption growth. Thus, total wealth is a function of both current and future expected consumption, and shocks to expected consumption growth translate into shocks to the realized return to wealth. This example illustrates how we can think of shocks to expected consumption growth as the second risk factor instead of the return to wealth.$^8$ Understanding the dynamic behavior of consumption growth is therefore necessary in order to understand the asset pricing properties of the production economy model with Epstein-Zin preferences. Before we consider different calibrations of the model, we therefore provide the general intuition for how consumption responds to both transitory and permanent technology shocks and how the consumption response relates to long-run risk.$^9$

$^8$ Following Bansal and Yaron (2004), we explicitly show this in the appendix through a log-linear approximation of the return to wealth.

$^9$ We make a strong distinction between transitory and permanent shocks in this section to provide clear intuition. As $\varphi \to 1$, the transitory shock specification (5) approaches the permanent shock specification (4). The dynamics of the model are in that case very similar for both specifications, so there is actually no
Figure 1 - Transitory and Permanent Shocks

Figure 1: **Impulse-Responses for Technology and Consumption.** Panel A shows the impulse-response of technology and consumption to a transitory technology shock. Panel B shows the impulse-response of technology and consumption to a permanent technology shock. The arrows show the direction in which the optimal consumption response changes if the desire for a smoother consumption path increases (i.e., the elasticity of intertemporal substitution decreases).

**Transitory Technology Shocks.** Panel A of figure 1 shows the impulse-response functions of technology and consumption to a transitory technology shock. Agents in this economy want to take advantage of the temporary increase in the productivity of capital due to the temporarily high level of technology. To do so, they invest immediately in capital at the expense of current consumption. As a result, the consumption response is hump-shaped. This figure illustrates how time-varying expected consumption growth arises endogenously in the production economy model: A positive shock to realized consumption growth (the initial consumption response) is associated with positive short-run expected consumption growth, but negative long-run expected consumption growth as consumption reverts back to the steady state. Thus, the shock to long-run expected consumption growth has the opposite sign of the shock to realized consumption growth, implying that shocks to realized consumption growth are hedged by shocks to the expected long-run consumption growth rate. As a consequence, long-run risk decreases the price of risk in the economy with transitory technology shocks.

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discontinuity at $\varphi = 1$ in terms of the model's asset pricing implications. However, the transitory shocks need to be extremely persistent for the transitory and permanent cases to be similar. At $\varphi = 0.9$, which is the case we consider in our calibration, the dynamic behavior of the model with permanent shocks is very different from the model with transitory shocks. The reader could therefore think of "transitory vs. permanent" shocks as "not extremely persistent vs. extremely persistent" shocks.
Permanent Technology Shocks. With permanent technology shocks, long-run consumption risk has the opposite effect. Panel B of figure 1 shows the impulse-response functions of technology and consumption to a permanent technology shock. Technology adjusts immediately to the new steady state, and the permanently higher productivity of capital implies that the optimal long-run levels of both capital and consumption are also higher. Agents invest immediately in order to build up capital at the expense of current consumption, and consumption gradually increases towards the new steady state after the initial shock. Thus, a positive shock to realized consumption growth (the initial consumption response) is associated with positive long-run expected consumption growth. In this case, long-run risk increases the price of risk in the economy because a positive technology shock induces positive shocks to both realized consumption growth and long-run expected consumption growth.

The Elasticity of Intertemporal Substitution. The elasticity of intertemporal substitution (EIS) is an important determinant of the dynamic behavior of consumption growth. A low EIS translates into a strong desire for intertemporally smooth consumption paths. In other words, agents strive to minimize the difference between their level of consumption today (after the shock) and future expected consumption levels. The arrows in figure 1 indicate the directions in which the initial optimal consumption responses change if the desire for a smoother consumption path increases. As the elasticity of intertemporal substitution decreases, agents desire a "flatter" response curve. From the figure, we can conjecture that a lower EIS decreases the volatility of expected future consumption growth. A high EIS, on the other hand, implies a higher willingness to substitute consumption today for higher future consumption levels. Therefore, the higher the EIS, the higher the volatility of expected consumption growth and the higher the levels of long-run risk in the economy. A high EIS thus decreases the price of risk if technology shocks are transitory, but increases the price of risk if technology shocks are permanent.

Capital Adjustment Costs. Capital adjustment costs (CAC) make it more costly for firms to adjust investment. Therefore, higher CAC induce lower investment volatility. We can therefore use CAC to match the relative volatilities of consumption, investment, and output with each model.
### Table 1

**Calibration**

Table 1: Calibrated values of parameters that are constant across models.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>Elasticity of capital</td>
<td>0.34</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Depreciation rate of capital</td>
<td>0.021</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Coefficient of relative risk aversion</td>
<td>5</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Mean technology growth rate</td>
<td>0.4%</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>Persistence of AR(1) technology</td>
<td>0.90</td>
</tr>
</tbody>
</table>

### 4 Results

The model generates macroeconomic aggregates such as output, investment, and consumption, in addition to the standard financial moments. In the first part of this section, we concentrate on the model’s ability to match key macroeconomic moments and on the dynamic behavior of consumption growth. Our discussion is centered around different values of the elasticity of intertemporal substitution and the two specifications of technology (permanent vs. transitory). We then discuss the asset pricing implications of the model and show that, for the permanent technology shock model, the parameters that best fit the macroeconomic moments are also the parameters that best fit the financial moments. In particular, we show how endogenous consumption smoothing and the long-run risk it generates can substantially improve the model’s ability to match the Sharpe ratio of equity returns. In this sense, we do not confirm Tallarini’s (2000) conclusion of a separation between preference parameters that govern the macroeconomic versus the financial moments. We solve the model numerically by means of the value function iteration algorithm. For a detailed discussion of our solution technique please refer to the appendix.

#### 4.1 Calibration

We report calibrated values of model parameters that are constant across models in Table 1. The capital share ($\alpha$), the depreciation rate ($\delta$), the mean technology growth rate ($\mu$), and the persistence of the transitory technology shocks ($\varphi$), are set to standard values for quarterly parameterizations (see, e.g., Boldrin, Christiano, Fisher, 2001). We set the coefficient of relative risk aversion ($\gamma$) to 5 across all models in the main part of the paper, while we use
different values for the elasticity of substitution ($\psi$) and for the time-discounting parameter ($\beta$). We also let the adjustment cost parameter ($\xi$), which denotes the elasticity of the investment to capital ratio with respect to Tobin’s $q$, and the conditional volatility of the technology shock ($\sigma_\varepsilon$) vary across models. We will discuss the choice of specific parameter values for each model below.

4.2 Macroeconomic Moments

In Table 2, we report relevant macroeconomic moments and consumption dynamics for models with either transitory or permanent technology shocks and different levels of the elasticity of intertemporal substitution ($\psi = 1/\gamma$, 0.5, 1.5).\(^\text{10}\) We match the U.S. output volatility over the period 1929 to 1998 with all models by setting the volatility of the technology shocks, $\sigma_\varepsilon$. We re-calibrate the discount factor ($\beta$) for each model so as to jointly match the values for ($C/Y$, $I/Y$, $D/Y$), that is aggregate average consumption, investment, and dividends relative to output, with each model. This is quite important, both since these are first-order moments and because we compare the volatility of growth rates across models. Capital adjustment costs ($\xi$) are the same across models and the value of $\xi$ is set in order to match the relative volatility of consumption to output with Model 6. The coefficient of relative risk aversion ($\gamma$) is constant across models. We show in the appendix, confirming Tallarini (2000), that the level of $\gamma$ has only second-order effects on the time series behavior of the macroeconomic variables.

4.2.1 The Volatility of Realized Consumption Growth

The volatility of realized consumption growth is the standard risk factor in consumption-based asset pricing models, where a higher volatility of consumption growth leads to a higher price of risk and a higher equity Sharpe ratio. Here, we discuss the endogenous volatility of realized consumption growth for the two technology specifications and different levels of the elasticity of intertemporal substitution ($EIS$).

In Models 1 to 3, technology shocks are transitory and the $EIS$ is increasing across models from the power utility case ($\psi = 1/\gamma = 0.2$, Model 1) to 1.5 (Model 3). Consumption volatility is increasing with $EIS$, while investment volatility is decreasing with the $EIS$. Agents with higher $EIS$ take advantage of a temporarily high technology level by consum-

\(^{10}\)In the transitory shock specification, log technology follows an AR(1) with a time trend. In the permanent shock specification log technology follows a random walk with drift. See equations (4) and (5).
Table 2

Macroeconomic Moments and Consumption Dynamics

Table 2: This table reports relevant macroeconomic moments and consumption dynamics for models with either transitory ($\varphi = 0.90$) or permanent technology shocks and different levels of the elasticity of intertemporal substitution ($\psi$). The coefficient of relative risk aversion ($\gamma$) is 5 across all models. We re-calibrate the discount factor ($\beta$) for each model so as to jointly match the values for (C/Y), (I/Y), (D/Y), with each model. Capital adjustment costs ($\xi$) are 30 in order to match the relative volatility of consumption to output with Model 6. We estimate the following process for the consumption dynamics:

$$c_{t+1} = \psi + \varepsilon_{t+1}$$
$$x_{t+1} = x_t + \varepsilon_{t+1}$$

$\psi$ denotes the standard deviation of variable $X$. We use annual U.S. data from 1929 to 1998 from the Bureau of Economic Analysis. The sample is the same as in Bansal and Yaron (2004). Under Panel B we report the calibration of the exogenous consumption process Bansal and Yaron use. All values reported in the table are quarterly.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
<th>Model 4</th>
<th>Model 5</th>
<th>Model 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transitory Shocks</td>
<td>Permanent Shocks</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_{\Delta y}$ (%)</td>
<td>2.62</td>
<td>2.62</td>
<td>2.62</td>
<td>2.62</td>
<td>2.62</td>
<td>2.62</td>
</tr>
<tr>
<td>$\sigma_{\Delta c}/\sigma_{\Delta y}$</td>
<td>0.52</td>
<td>0.28</td>
<td>0.32</td>
<td>0.36</td>
<td>1.01</td>
<td>0.83</td>
</tr>
<tr>
<td>$\sigma_{\Delta i}/\sigma_{\Delta y}$</td>
<td>3.32</td>
<td>4.02</td>
<td>3.77</td>
<td>3.68</td>
<td>0.98</td>
<td>1.68</td>
</tr>
<tr>
<td>$E[C/Y]$</td>
<td>0.79</td>
<td>0.79</td>
<td>0.79</td>
<td>0.79</td>
<td>0.79</td>
<td>0.79</td>
</tr>
</tbody>
</table>

Panel A: Macroeconomic Moments (Quarterly)

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
<th>Model 4</th>
<th>Model 5</th>
<th>Model 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bansal, Yaron Calibration</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_{\Delta c}$ (%)</td>
<td>1.360</td>
<td>0.734</td>
<td>0.838</td>
<td>0.943</td>
<td>2.635</td>
<td>2.176</td>
</tr>
<tr>
<td>$\sigma_{\varepsilon}$ (%)</td>
<td>1.351</td>
<td>0.734</td>
<td>0.831</td>
<td>0.926</td>
<td>2.631</td>
<td>2.166</td>
</tr>
<tr>
<td>$\sigma_{x}$ (%)</td>
<td>0.172</td>
<td>0.066</td>
<td>0.100</td>
<td>0.170</td>
<td>0.114</td>
<td>0.209</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.938</td>
<td>0.986</td>
<td>0.968</td>
<td>0.922</td>
<td>0.983</td>
<td>0.972</td>
</tr>
<tr>
<td>$\sigma_{\varepsilon}$ (%)</td>
<td>0.059</td>
<td>0.011</td>
<td>0.025</td>
<td>0.066</td>
<td>0.021</td>
<td>0.049</td>
</tr>
</tbody>
</table>
ing relatively more today and less in the future as technology reverts back to its long-run trend. As a result, the level of risk associated with shocks to realized consumption growth is increasing with the EIS in the model with transitory shocks.

In Models 4 to 6, the technology shocks are permanent. Here the consumption growth volatility is decreasing with the EIS. Consider a positive shock to technology. Since the shock is permanent, agents with a high EIS want to increase the capital stock to its new optimal level as quickly as possible for consumption to grow faster towards its new, permanently higher level. To that end they need to invest more today, implying a smaller initial consumption response. Thus, the level of risk associated with shocks to realized consumption growth is decreasing with the EIS in the model with permanent shocks. With respect to this standard risk factor, a higher EIS therefore reduces risk in the permanent shock model.

In our model, however, shocks to expected consumption growth are also a risk factor and consequentially we now turn to the dynamic behavior of expected consumption growth.

4.2.2 The Volatility of Expected Consumption Growth (Long-run Risk)

Our model does not allow analytical expressions for the dynamic behavior of consumption growth. To nonetheless get a hand on the endogenous consumption dynamics, we estimate the following approximate system from simulated data of our model:

\[ \Delta c_{t+1} = \mu + x_t + \sigma_{\eta} \eta_{t+1}, \]  
\[ x_{t+1} = \rho x_t + \sigma_{e} e_{t+1}, \]  
\[ \sigma_{\eta,e} = \text{corr} \left( \eta_{t+1}, e_{t+1} \right). \]

Here \( \Delta c_{t+1} \) is log realized consumption growth, \( x_t \) is the time-varying component of expected consumption growth, and \( \eta_t, e_t \) are zero mean, unit variance disturbance terms with correlation \( \sigma_{\eta,e} \). This functional form for log consumption growth is identical to the one assumed by Bansal and Yaron (2004) as the driving process of the first exchange economy model in their paper, which allows us to quantitatively compare our results to theirs. In the appendix, we show that the above system is a surprisingly good approximation for models where log technology follows a random walk.\(^{11}\)

---

\(^{11}\)For models where log technology follows an AR(1) the above approximation of the endogenous consumption dynamics is less good, which is to be expected given the shape of the impulse-response function of consumption in that case (see Figure 1). There is also some heteroskedasticity in both shocks to expected and realized consumption growth.
Panel B of table 2 shows the estimated parameters for each of the 6 models, as well as the parameters Bansal and Yaron (2004) use. Comparing the relative magnitudes of the volatility of realized consumption growth and the shock to expected consumption growth ($\sigma_n$ vs. $\sigma_x$), we can see that the time-varying growth component is very small. The implied average $R^2$ across models is around $1-2\%$. Note however that the persistence of the expected consumption growth rate ($\rho$) is very high, which is important if risk associated with a small time-varying expected consumption growth rate component is to have quantitatively interesting asset pricing implications. As expected from the discussion in section 3, the volatility of expected consumption growth, $\sigma[x]$, is increasing in the elasticity of intertemporal substitution, that is long-run risk is increasing in the $EIS$. Whether this risk factor increases or decreases the price of risk in the economy depends on the net effect on the return to total wealth and its correlation with realized consumption growth.

Before we get to the asset pricing moments, however, we note that the model implies observable proxies for expected consumption growth. This is a nice implication of the model as expected consumption growth otherwise is a latent variable that it is very hard to measure (see, e.g., Harvey and Shepard, 1990, and Hansen, Heaton and Li, 2005).

4.2.3 Observable Proxies for Expected Consumption Growth

An empirical obstacle to using Epstein-Zin preferences is that the return on total wealth shows up in the stochastic discount factor (see eq. (37)). This quantity is unobservable. Following Bansal and Yaron (2004), we show in section 5.2 that shocks to expected consumption growth and the current level of expected consumption growth can be used as factors instead of the return to wealth. Unfortunately, these factors are also unobservable. The production economy model, however, connects technology, consumption, and investment. We can take advantage of this structural link to identify observable proxies for the otherwise unobservable expected consumption growth rate.

Figure 2 shows the impulse-response of consumption to a one standard deviation permanent shock to technology (total factor productivity) for high and low levels of the $EIS$. With higher $EIS$ the initial consumption response is lower as the consumer is happy to substitute consumption today for more consumption tomorrow. The plot suggests that the ratio of total factor productivity (TFP) to consumption is a good proxy for the current level of expected consumption growth. In particular, when consumption is low relative to TFP, future consumption growth is expected to be high and vice versa. Data on total factor pro-

18
ductivity and consumption are both readily available. We can therefore use the log TFP to consumption ratio as a proxy for expected consumption growth. We verify this result in the empirical part of the paper (section 5).

![Aggregate Consumption Graph](image1)

**Figure 2:** Impulse Responses of Consumption and Investment  Impulse responses of consumption and investment to a one standard deviation positive and permanent shock to technology for different levels of the EIS. The impulse responses are for Model 5 (EIS = 0.5) and Model 6 (EIS = 1.5) respectively.

### 4.3 Asset Pricing Implications

Table 3 shows important financial moments for a range of different models. The data are taken from Bansal and Yaron (2004) who use annual U.S. data from 1929 to 1998. We calibrate the volatility of aggregate consumption growth to its empirical counterpart for each model we report in Table 3 by adjusting the volatility of technology growth. Keeping the volatility of aggregate consumption growth constant across models allows us best to compare asset prices. We use the coefficient of relative risk aversion ($\gamma$), the discount factor ($\beta$), and adjustment costs ($\xi$) to respectively match the equity Sharpe ratio, the level of
Table 3

**Financial Moments**

Table 3: This table reports relevant financial moments and consumption dynamics for models with either transitory ($\varphi = 0.90$) or permanent technology shocks and different levels of the elasticity of intertemporal substitution ($\psi$). The coefficient of relative risk aversion ($\gamma$) is 5 across all models, while the discount factor ($\beta$) is 0.998 and capital adjustment costs ($\xi$) are 22, in order to match the equity Sharpe ratio, the level of the risk-free rate, and the relative volatility of consumption to output with Model 12. We re-calibrate $\sigma_e$ in order to match the volatility of consumption growth with each model. We estimate the following process for the consumption dynamics:

$$c_{t+1} = \mu + c_t + \sigma_\eta \eta_{t+1}, \quad x_{t+1} = \rho x_t + \sigma_e \epsilon_{t+1}.$$

$\sigma[X]$ denotes the standard deviation of variable $X$. The data are taken from Bansal and Yaron (2004) who use annual U.S. data from 1929 to 1998. All values reported in the table are annual.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Model 7</th>
<th>Model 8</th>
<th>Model 9</th>
<th>Model 10</th>
<th>Model 11</th>
<th>Model 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma[\Delta c]$ (%)</td>
<td>2.72</td>
<td>2.72</td>
<td>2.72</td>
<td>2.72</td>
<td>2.72</td>
<td>2.72</td>
</tr>
<tr>
<td>$\sigma[x]$ (%)</td>
<td>n/a</td>
<td>0.12</td>
<td>0.14</td>
<td>0.19</td>
<td>0.06</td>
<td>0.10</td>
</tr>
<tr>
<td>$\sigma[M]/E[M]$</td>
<td>n/a</td>
<td>0.13</td>
<td>0.08</td>
<td>0.02</td>
<td>0.13</td>
<td>0.18</td>
</tr>
<tr>
<td>$SR[r^A]$</td>
<td>n/a</td>
<td>0.08</td>
<td>0.05</td>
<td>0.02</td>
<td>0.12</td>
<td>0.16</td>
</tr>
<tr>
<td>$SR[r^E]$</td>
<td>0.33</td>
<td>0.10</td>
<td>0.07</td>
<td>0.02</td>
<td>0.13</td>
<td>0.18</td>
</tr>
</tbody>
</table>

**Panel A: The Price of Risk and Consumption Dynamics (Annual)**

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Model 7</th>
<th>Model 8</th>
<th>Model 9</th>
<th>Model 10</th>
<th>Model 11</th>
<th>Model 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[r_f]$ (%)</td>
<td>0.86</td>
<td>7.55</td>
<td>3.68</td>
<td>1.80</td>
<td>7.60</td>
<td>3.43</td>
</tr>
<tr>
<td>$\sigma[r_f]$ (%)</td>
<td>0.97</td>
<td>1.12</td>
<td>0.55</td>
<td>0.25</td>
<td>0.57</td>
<td>0.41</td>
</tr>
<tr>
<td>$E[r^A - r_f]$ (%)</td>
<td>n/a</td>
<td>0.54</td>
<td>0.25</td>
<td>0.03</td>
<td>0.02</td>
<td>0.18</td>
</tr>
<tr>
<td>$\sigma[r^A - r_f]$ (%)</td>
<td>n/a</td>
<td>6.60</td>
<td>4.67</td>
<td>1.84</td>
<td>0.16</td>
<td>1.13</td>
</tr>
<tr>
<td>$E[r^E - r_f]$ (%)</td>
<td>6.33</td>
<td>0.22</td>
<td>0.12</td>
<td>0.02</td>
<td>0.03</td>
<td>0.05</td>
</tr>
<tr>
<td>$\sigma[r^E - r_f]$ (%)</td>
<td>19.42</td>
<td>2.32</td>
<td>1.77</td>
<td>1.29</td>
<td>0.22</td>
<td>0.28</td>
</tr>
</tbody>
</table>
the risk-free rate, as well as the relative volatility of consumption to output with Model 12, which is the model that best fits the macroeconomic moments. We keep these parameters \((\gamma, \beta, \xi)\) constant across models in order to examine the effect of the \textit{EIS} and technology specification on endogenous long-run risk and asset prices.

### 4.3.1 The Price of Risk and Sharpe Ratios

Panel A of Table 3 shows that long-run risk, as measured by the volatility of the expected consumption growth rate, \(\sigma [x]\), increases substantially for both the case of permanent and transitory shocks as we increase the \textit{EIS}. However, the price of risk \((\sigma [M]/E [M])\) and the Sharpe ratio of asset- and equity returns are \textit{decreasing} with transitory shocks and \textit{increasing} with permanent shocks. As discussed in section 3, with transitory shocks the long-run response of consumption mean-reverts back to the steady state. Thus, a positive shock to realized consumption growth is associated with a negative shock to expected long-run consumption growth. The long-run risk is a hedge for the shock to realized consumption growth, and the table makes clear that this effect is quantitatively important: The Sharpe ratio of equity returns drops from 0.10 to 0.02 as the \textit{EIS} increases from 0.2 (= 1/\(\gamma\)) to 1.5. In the case of permanent shocks, we see the opposite effect. Here, the Sharpe ratio of equity returns increases from 0.13 to 0.33 as the \textit{EIS} increases from 0.2 to 1.5. With permanent shocks, long-run risk renders the economy riskier for consumers who dislike negative shocks to future growth prospects: Comparing Model 10 (power utility) with Model 12 \((\gamma > \frac{1}{\psi})\), endogenous long-run risk combined with a preference for early resolution of uncertainty almost triples the price of risk in the economy.

We can therefore conclude that, given a random walk as the stochastic driver of the economy, the asset pricing properties of the standard production economy model are actually \textit{better} than the properties of the standard exchange economy model, contrary to what is usually taken for granted in the literature. Note that the price of risk of the exchange economy counterpart of Model 12 with consumption growth specified as a random walk is 0.13.\footnote{The volatility of the stochastic discount factor when the representative agent has power utility is approximately \(\gamma \sigma_{Dc} = 0.13\). In Model 12, the price of risk is in fact 0.33 - almost three times as large.}
4.3.2 The Risk-free Rate

Panel B of Table 3 shows that the risk-free rate is decreasing in the $EIS$, as expected. A higher $EIS$ decreases the intertemporal substitution effect (see eq. (38)), and the volatility of the risk-free rate is low. The time-variation in expected consumption growth rates does not induce too volatile risk-free rates, because the growth shocks are very persistent and not very volatile. This is an improvement over habit formation models like in Jermann (1998) or Boldrin, Christiano and Fisher (2001), where time-variation in the state variable "surplus consumption" induces much too volatile risk-free rates.

4.3.3 The Return Premium and Return Volatility

For the models with a permanent shock, both the total asset premium as well as the equity premium are strongly increasing in the $EIS$ as both the Sharpe ratio and the volatility of returns are increasing in the $EIS$. From the power utility model with $\gamma = 5$, $\psi = \frac{1}{\gamma}$ to the case with $\gamma = 5$, $\psi = 1.5$ the equity premium increases six-fold from 0.03% to 0.19%, and equity return volatility increases from 0.22% to 0.57%. While these are substantial relative increases, the equity premium is still roughly a magnitude too low due to the low volatility of returns to the equity claim. Note, however, that the asset return premium and the asset return volatility are much more sensitive to increases in the $EIS$. The reason is the difference of the respective dividend processes. While equity dividends are given by $D^E_t = \alpha Y_t - I_t$, total asset dividends are given by $D^A_t = C_t = Y_t - I_t$. Consider a permanent, positive shock to technology. If investors have higher $EIS$, this results in higher investment volatility and higher expected future consumption growth. Both equity dividends as well as total asset dividends respond now less to a positive shock. However, equity dividends are much more sensitive to this effect, and may even decrease in response to a shock, implying a negative correlation between dividend growth and expected consumption growth. The result is that the equity return volatility, and thus the equity premium, increase by less with the $EIS$ relative to the total asset return. Bansal and Yaron (2004) exploit the fact that the asset and equity claims have different dividend processes (i.e., consumption and dividends), and use this as a degree of freedom to fit the asset price moments. In their model, the dividend process is exogenously specified and expected dividend growth is assumed to be very sensitive to shocks to expected consumption growth, but dividends are not cointegrated with consumption. If we increase the $EIS$ in the Bansal and Yaron framework, the dividend response is unaffected. That way, Bansal and Yaron are able to fit the equity volatility,
and thus the equity premium, with roughly the same (exogenous) consumption process and preference parameters as in our models. The production economy model, on the other hand, puts tight restrictions on the joint dynamic behavior of aggregate consumption and dividends. Bansal and Yaron assume this correlation to be zero. If we set this correlation to $-1$ in their model, the annual volatility of equity returns drops from 16% to less than 1%.\footnote{This is for the model with no time-varying volatility of consumption growth rates, $\gamma = 7.5$ and $\psi = 1.5$.} Thus, while the production economy model provides a theoretical justification for a consumption process with long-run risk, it does not give a justification for modeling dividends as essentially unrelated to consumption over the long-run. The general equilibrium framework imposes constraints that are unfavorable in terms of matching the volatility of equity returns.

4.3.4 Conditional Moments of the Equity Return

The model has interesting implications for the conditional moments of the equity return, induced by time-varying second moments of the consumption and dividend processes. In particular, figure 3 shows that expected excess returns and volatility of our benchmark Model 12 are counter-cyclical. The moments are plotted against a measure of the state of the economy - expected consumption growth, which is high in expansions and low in recessions.\footnote{Expected consumption growth has a one-to-one mapping to the state variable of the model, capital normalized by the level of technology. A series of positive technology shocks means times are "good", productivity of capital is high, and investment is high, and so is expected consumption growth.} The Sharpe ratio of equities, however, is slightly pro-cyclical, although this effect is tiny compared to the level of the Sharpe ratio. The slight change in Sharpe ratio is due to slight heteroskedasticity in shocks to both realized and expected consumption growth. The net effect of this on the price of risk and the equity Sharpe ratio, however, is very small.

The counter-cyclical risk premium is driven by the counter-cyclical volatility of equity returns. Both features are in line with the stylized facts on stock market returns. Equity volatility is counter-cyclical for the same reason that asset returns are unconditionally more volatile than equity returns - the different dynamics of dividends and consumption. Since $C_t = Y_t - I_t$ and $D_t = \alpha Y_t - I_t$, differences in the dynamic behavior of dividends and consumption are driven by investment. In a recession, the level of investment is low and therefore the dividend and consumption processes are more similar. The final panel of figure 3 confirms this intuition by showing the conditional covariance of consumption and dividends, which is strongly counter-cyclical. In other words, consumption and dividends co-vary more during recessions, inducing more volatile equity returns. Thus, the production
Figure 3: Conditional Moments of the Equity Return

The figure shows expected excess annual return, volatility and Sharpe ratio of equity returns as a function of expected consumption growth. The moments are from Model 12 (EIS = 1.5 and permanent technology shocks). The last panel shows the conditional covariance of dividends and consumption, normalized by the level of technology.

4.4 The Case of $\gamma < \frac{1}{\psi}$

In the previous discussion our focus was on the case of $\gamma > \frac{1}{\psi}$ where agents prefer early resolution of uncertainty and dislike fluctuations in expected consumption growth rates. With transitory shocks and $\gamma > \frac{1}{\psi}$, this second risk factor acts as a hedge for shocks to realized consumption growth and therefore reduces the price of risk. This raises the possibility that if agents like fluctuations in expected consumption growth rates, that is when $\gamma < \frac{1}{\psi}$,
consumption smoothing increases the price of risk when technology shocks are transitory. In this section, we investigate whether this channel can give rise to long-run risks that help in explaining asset prices with a low elasticity of intertemporal substitution. The short answer is yes. However, a low EIS unfortunately gives rise to a risk-free rate puzzle (Weil, 1989).

Table 4 shows calibrated macroeconomic and financial moments from models with $\gamma = 5$, and $\psi = \frac{1}{5}$ and $\psi = 0.1$. From Model 7 to Model 13 we increase capital adjustment costs in order to fit the relative volatility of consumption to output. Higher capital adjustment costs increase the equity return volatility, as marginal $q$ is now more volatile. From Model 13 to Model 14 we decrease the EIS relative to the benchmark power utility model. A lower EIS induces a higher equity return Sharpe ratio and a higher price of risk due to the preference for time-varying expected consumption growth, which is negatively correlated with shocks to realized consumption growth. In other words, the effect of long-run risk on the price of risk is now decreasing in the EIS, as opposed to the case of permanent technology shocks. Long-run risk with a low EIS is quite different from the intuition in Bansal and Yaron (2004) and may seem surprising given that a low EIS implies that consumers strive to make the consumption path smooth over time and therefore minimize the volatility of expected consumption growth. However, the fact that we are increasing capital adjustment costs to match the macroeconomic moments renders consumption smoothing more costly. Therefore, in equilibrium, a substantial amount of long-run risk remains in the economy even with a low EIS. As a result, the price of risk almost doubles relative to the power utility benchmark model. The equity return volatility also increases due to high capital adjustment costs. The net effect is a substantial increase in the equity premium.

Unfortunately, decreasing the elasticity of intertemporal substitution also leads to a risk-free rate puzzle. The average annual risk-free rate is with 7.5% already much too high in the benchmark power utility model and increases further to 14.6% as we decrease the EIS. Thus, endogenous long-run risk can substantially improve the asset pricing properties of production economy models even with a very low EIS. However, the risk-free rate is too high given an average annual consumption growth rate of 1.6% as assumed in this paper. If one is willing to assume that the average real consumption growth rate is close to zero, it would be possible to also fit the average risk-free rate.
Table 4: The Case of $\gamma < \frac{1}{\psi}$: Asset Pricing Implications

Table 4: This table reports relevant macroeconomic moments, consumption dynamics, and financial moments for models with transitory ($\varphi = 0.90$) technology shocks and different levels of the elasticity of intertemporal substitution. The coefficient of relative risk aversion ($\gamma$) is 5 across all models. We re-calibrate $\sigma_t$ in order to match the volatility of consumption growth with each model. We estimate the following process for the consumption dynamics: $\Delta c_{t+1} = \mu + x_t + \sigma \eta_{t+1}$, $x_{t+1} = \rho x_t + \sigma e_{t+1}$. $\Delta x = \log(X_t) - \log(X_{t-1})$, and $\sigma[X]$ denotes the standard deviation of variable $X$. We use annual U.S. data from 1929 to 1998 from the Bureau of Economic Analysis and from Bansal and Yaron (2004). Under Panel B we report the calibration of the exogenous consumption process Bansal and Yaron use. All values reported in Panels A and B are quarterly, all values reported in Panel C are annual.

<table>
<thead>
<tr>
<th>Model 7</th>
<th>Model 13</th>
<th>Model 14</th>
<th>Model 15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transitory Shocks $z_{t+1} = \mu t + \varphi z_t + \sigma \varepsilon_{t+1}$</td>
<td>$\psi = \frac{1}{7}$</td>
<td>$\psi = \frac{1}{7}$</td>
<td>$\psi = 0.10$</td>
</tr>
<tr>
<td>$\xi = 22$</td>
<td>$\xi = 2$</td>
<td>$\xi = 2$</td>
<td>$\xi = 2$</td>
</tr>
<tr>
<td>Statistic $\beta = 0.998$</td>
<td>$\beta = 0.998$</td>
<td>$\beta = 0.998$</td>
<td>$\beta = 0.99999$</td>
</tr>
</tbody>
</table>

Panel A: Macroeconomic Moments (Quarterly)

<table>
<thead>
<tr>
<th>U.S. Data 1929-1998</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma[\Delta y]$ (%)</td>
</tr>
<tr>
<td>$\sigma[\Delta c]/\sigma[\Delta y]$</td>
</tr>
<tr>
<td>$\sigma[\Delta i]/\sigma[\Delta y]$</td>
</tr>
</tbody>
</table>

Panel B: Consumption Dynamics (Quarterly): $\Delta c_{t+1} = \mu + x_t + \sigma \eta_{t+1}$, $x_{t+1} = \rho x_t + \sigma e_{t+1}$.

<table>
<thead>
<tr>
<th>Bansal, Yaron Calibration</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma[\Delta c]$ (%)</td>
</tr>
<tr>
<td>$\sigma[x]$ (%)</td>
</tr>
<tr>
<td>$\rho$</td>
</tr>
<tr>
<td>$\sigma_e$ (%)</td>
</tr>
</tbody>
</table>

Panel B: Financial Moments (Annual)

| $\sigma [M]/E[M]$ | n/a | 0.13 | 0.13 | 0.19 | 0.19 |
| $SR [r^E]$ | 0.33 | 0.10 | 0.10 | 0.14 | 0.14 |
| $E[r_f] (%)$ | 0.86 | 7.55 | 7.52 | 14.60 | 13.91 |
| $\sigma[r_f]$ (%) | 0.97 | 1.12 | 1.92 | 3.17 | 3.10 |
| $E[r^E - r_f]$ (%) | 6.33 | 0.22 | 0.74 | 1.44 | 1.41 |
| $\sigma[r^E - r_f]$ (%) | 19.42 | 2.32 | 7.59 | 10.38 | 10.28 |

26
5 Expected Consumption Growth and the Cross-Section of Stock Returns

We test some of our model’s predictions both for the time series of technology (total factor productivity) and consumption growth as well as for the cross-section of stock returns. In particular, we are interested whether proxies for expected consumption growth suggested by our model actually forecast long-horizon consumption growth or not, whether shocks to expected consumption growth are a priced risk factor or not, and whether the price of risk is positive or negative.

The consumption and TFP data are from the Bureau of Economic Analysis and the Bureau of Labor Statistics respectively, the return data are from Kenneth French’s homepage (post-war sample).\footnote{http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html}

5.1 Expected Consumption Growth

We have demonstrated that the system

\[
\begin{align*}
\Delta c_{t+1} &= \mu + x_t + \sigma_n e_{t+1}, \\
x_{t+1} &= \rho x_t + \sigma_e e_{t+1}, \\
\sigma_{n,e} &= corr(\eta_{t+1}, e_{t+1}),
\end{align*}
\]

is a good, parsimonious specification for the consumption growth process generated by our model. As highlighted by Bansal and Yaron (2004) and Hansen, Heaton and Li (2005), amongst others, it is very hard to estimate long-run consumption growth dynamics from the relatively short samples of data we have available. In our model, slow-moving expected consumption growth dynamics arise due to endogenous consumption smoothing. Investors respond to a technology shock by increasing investment in order to build up higher levels of capital. Out of the higher capital stock investors gradually increase consumption towards the new steady state. Thus, while technology immediately adjusts to its new permanent level, consumption slowly grows to a permanently higher level. It follows that our model implies the difference between the aggregate technology level and the current level of consumption
to be a good instrument for the expected consumption growth rate. We define

$$zc_t \equiv \ln \left( \frac{Z_t}{C_t} \right).$$

(18)

If $zc_t$ is high, consumption is more likely to increase towards a new steady-state level. Our model thus implies $zc_t$ to be a good instrument for the expected consumption growth rate $x_t$. Our model predicts that $zc_t = \ln \left( \frac{Z_t}{C_t} \right)$ is stationary even though both $Z_t$ and $C_t$ are non-stationary ($I(1)$), because (i) $Z_t$ and $C_t$ are cointegrated and (ii) $C_t$ evolves around the stochastic trend $Z_t$. It is well-known that if the production technology is specified as:

$$Y_t = Z_t^{1-\alpha} K_t^\alpha N_t^{1-\alpha},$$

(19)
as is the case in our model, all endogenous variables in the economy evolve around the stochastic trend $Z_t$ (see appendix 8.4). We get data on $Z_t$ (TFP) from the Bureau of Labor Statistics (BLS). The BLS computes TFP as follows. First it collects data on $Y_t$ (output), on $K_t$ (capital input), and on $N_t$ (labor input). Then the BLS estimates a value for the parameter $\alpha$ and computes TFP as the Solow residual:

$$\ln \tilde{Z}_t = \ln Y_t - \alpha \ln K_t - (1 - \alpha) \ln N_t.$$

(20)

Note that the BLS specifies the following production technology:

$$Y_t = \tilde{Z}_t K_t^\alpha N_t^{1-\alpha}.$$

(21)

It follows that we need to normalize:

$$Z_t = \tilde{Z}_t^{1/(1-\alpha)}.$$  

(22)

We take as the value for $\alpha$ the value we use in our model ($\alpha = 0.34$). We check our results for robustness by assuming different values for $\alpha \in [0.30, 0.40]$, and find that our results are robust with respect to the choice of $\alpha$.

The benchmark Model 12, which has permanent technology shocks, suggests the following forecasting relationship:

$$\Delta c_{t,t+j} = \alpha + \beta z c_t + \varepsilon_{t,t+j},$$

(23)
where $\beta > 0$. In the model, the relation is not exactly linear, but when simulating data from our model (Model 12) we found that $z_{ct}$ accounts for more than 99% of the variation in $x_t$ in a linear regression. Because $z_{ct}$ is a proxy for the slow-moving expected consumption growth rate, it forecasts future consumption growth. We test this forecasting relationship both on data from 1948 to 2005 and on data generated by our model (Model 12). In particular, in Table 5 we report results from forecasting regressions of annual log nondurable- and services consumption growth on the lagged log TFP to consumption ratio, our measure of expected consumption growth.

Panel A shows that consumption growth is forecastable by the $z_{c-ratio}$ using simulated data from our model. The regression coefficient is increasing in the horizon up to 7 years. The forecasting regression coefficients are found by simulating 10,000 samples of length 58 years, running the regression on each sample, and computing the average regression coefficient. The sample errors are the sample standard deviation of each $\beta$-estimate. Interestingly, the regression coefficient is not significant in any of the regressions. The variation in expected consumption growth is too slow-moving for the regressions to on average uncover the forecasting relationship over the relatively short sample period. Panel B shows the results from the forecasting regressions using real data. Here both the regression coefficients and the $R^2$'s are increasing with the forecasting horizon. The coefficients are significant at the 10% level for all regressions using Hodrick (1992) standard errors, which have relatively good small sample properties for overlapping regressions. The coefficients are overall lower than those estimated using simulated data. This could be because there is less variation in expected consumption growth in the data or because the empirical $z_{c-ratio}$ is measured with noise. The $R^2$'s are higher for the long-horizon regressions than predicted by the model. However, Valkanov (2003) shows that $R^2$'s are badly behaved in small samples where the fraction of overlapping observations relative to the total sample length is large. In particular, the sample $R^2$'s are likely to overstate the true $R^2$'s. We conclude that the log TFP to consumption ratio, a measure of expected consumption growth implied by our theoretical model, forecasts future consumption growth and that the level of measured variation in expected consumption growth is similar to that implied by our model with $\gamma = 5$ and $\psi = 1.5$.

### 5.2 The Cross-Section of Stock Returns

The model in this paper implies that the shock to expected consumption growth is a priced risk factor as long as $\gamma \neq \frac{1}{\psi}$, i.e. as long as agents care about the temporal resolution of risk.
### Table 5

**Estimating Expected Consumption Growth**

Table 5: This table reports forecasting regressions of annual log nondurable- and services consump-
tion growth on a lagged measure of expected consumption growth, the log TFP to Consumption
ratio. The consumption and TFP data are from the Bureau of Economic Analysis and the Bureau
of Labor Statistics respectively. We use annual data from 1948 to 2005, resulting in 58 - $j$ obser-
vations for a regression with a $j$ year forecasting horizon: Multi-year forecasting regressions are
overlapping at an annual frequency. The standard error estimates (in parenthesis) are corrected for
heteroskedasticity and overlapping observations using Hodrick (1992) standard errors. Results for
the model are based on 10,000 replications of sample size 58 × 4 each. Numbers in bold indicate
significance at the 5% level or more in a two-tailed t-test, while an asterisk indicates significance
at the 10% level.

#### Regression: $\Delta c_{t,t+j} = \alpha + \beta z_{c_t} + \epsilon_{t,t+j}$

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.077</td>
<td>0.135</td>
<td>0.178</td>
<td>0.208</td>
<td>0.226</td>
<td>0.233</td>
<td>0.192</td>
</tr>
<tr>
<td></td>
<td>(0.051)</td>
<td>(0.101)</td>
<td>(0.151)</td>
<td>(0.201)</td>
<td>(0.251)</td>
<td>(0.345)</td>
<td>(0.468)</td>
</tr>
<tr>
<td>$R^2_{adj}$</td>
<td>9.2%</td>
<td>10.2%</td>
<td>10.9%</td>
<td>10.8%</td>
<td>10.3%</td>
<td>9.3%</td>
<td>8.3%</td>
</tr>
<tr>
<td></td>
<td>(0.092)</td>
<td>(0.118)</td>
<td>(0.128)</td>
<td>(0.131)</td>
<td>(0.131)</td>
<td>(0.128)</td>
<td>(0.124)</td>
</tr>
</tbody>
</table>

#### Panel A: Model implied ($j$ denotes forecasting horizon in years)

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.021*</td>
<td>0.041*</td>
<td>0.060*</td>
<td>0.084*</td>
<td>0.107*</td>
<td>0.147*</td>
<td>0.233*</td>
</tr>
<tr>
<td></td>
<td>(0.012)</td>
<td>(0.024)</td>
<td>(0.036)</td>
<td>(0.048)</td>
<td>(0.060)</td>
<td>(0.085)</td>
<td>(0.127)</td>
</tr>
<tr>
<td>$R^2_{adj}$</td>
<td>3.3%</td>
<td>5.5%</td>
<td>7.6%</td>
<td>11.1%</td>
<td>13.8%</td>
<td>17.6%</td>
<td>30.0%</td>
</tr>
</tbody>
</table>

### Panel B: Historical estimates ($j$ denotes forecasting horizon in years)
The cross-section of stock returns can tell us both whether shocks to expected consumption growth are a priced risk factor and whether the price of risk on this factor is positive or negative, which in turn depends on whether the relative risk aversion of the representative agent is smaller or larger than the reciprocal of the elasticity of substitution and on the persistence of the technology shocks. Given the consumption dynamics in (15) and log-linearizing the return on the wealth portfolio around the steady state ratio of wealth to aggregate consumption, the stochastic discount factor can be written as:

\[ m_{t+1} \approx a - b_1 \Delta c_{t+1} - b_2 e_{t+1} - b_3 x_t, \]  

(24)

where \( \Delta c_{t+1} \) denotes realized consumption growth, \( x_t \) is the current level of expected consumption growth, \( e_{t+1} \) is the shock to expected consumption growth, and \( b_1 = \gamma, b_2 = (1 - \theta) A_1 \kappa_1 \sigma_x, b_3 = (\theta - 1) A_1 (1 - \kappa_1 \rho) \) (see appendix for a detailed derivation and definitions of the constants \( A_1, \kappa_1 > 0 \)). If \( \gamma > \frac{1}{\psi} \), the coefficients \( b_1, b_2 > 0 \) and \( b_3 < 0 \).

Based on results derived in the previous section, we use the variable \( zc_t \) to obtain measures of \( x_t \) and \( e_{t+1} \):

\[ \begin{align*}
\hat{x}_t &= k_0 + k_1 zc_t, \\
\hat{e}_{t+1} &= k_1 \left[ zc_{t+1} - \hat{\mu}_{zc} - \hat{\phi} zc_t \right].
\end{align*} \]

(25)  

(26)

where \( k_1 > 0 \).

By applying a standard log-linear approximation of the stochastic discount factor (see appendix), we arrive at the linear factor model:

\[ E[R_{i,t} - R_{0,t}] = \beta_i^T \lambda, \]

(27)

where \( \beta_i \) is the vector of regression coefficients for asset \( i \) on the three factors, and \( \lambda \) is the vector of risk prices for the factors. From the approximating equation of the stochastic discount factor (24) we have that \( \lambda_{\Delta c} > 0, \lambda_x < 0, \lambda_e > 0 \) if \( \gamma > \frac{1}{\psi} \) (see appendix). In other words, there is a positive price of risk for exposure to shocks to realized consumption growth as well as for exposure to shocks to expected consumption growth. The standard consumption-based asset pricing model with power utility implies that \( b_1 = \gamma = \frac{1}{\psi}, \) while \( b_2 = b_3 = 0 \). The second risk factor, the shock to expected consumption growth, has a positive price of risk as long as \( \gamma > \frac{1}{\psi} \). The current level of expected consumption growth (\( x_t \)) has a negative price of risk. The unconditional covariance of returns with total consumption
growth will overstate the covariance of returns with innovations to consumption growth if \( \text{Cov}(x_t, R_{i,t+1} - R_{0,t+1}) > 0 \) and induce too high expected excess returns. The term \( \beta_{t,x} \lambda_x \) corrects for this effect and therefore \( \lambda_x < 0 \) (see appendix).

Because TFP data are only available on an annual basis from the Bureau of Economic Analysis, we use annual returns on the 25 Fama-French portfolios as test assets along with the risk-free rate (U.S. t-Bill). The sample consists of 57 observations from 1948 – 2005. Table 6 displays results for the benchmark Consumption CAPM \( (b_2 = b_3 = 0) \) and the three-factor model of this paper, as laid out above.

Table 6 displays the sign of the estimated quantities with p-values in parentheses. The price of realized consumption growth risk is insignificant in the standard Consumption CAPM model and the adjusted \( R^2 \) is 17.4%. The three-factor model including measures of the level and the shock to expected consumption growth increases the adjusted \( R^2 \) to 46.6%. The price of risk for the shock to expected consumption growth carries a positive sign and is significant at the 10% level. The sign on the measure of expected consumption growth is negative, but not significant. We conclude that the shock to the expected consumption growth rate is a priced risk factor. The positive sign is consistent with a model where technology shocks are permanent and agents prefer early resolution of uncertainty \( (\gamma > \gamma^*) \), and a model where technology shocks are transitory and \( \gamma < \gamma^* \).

As emphasized by Cochrane (2001), if a "new" factor carries a positive risk premium this does not necessarily imply that the new factor really helps with pricing assets relative to a benchmark model with an "old" factor. The new factor could simply be correlated with the old risk factor. Taking the standard CCAPM as the benchmark model, the significance of the shock to expected consumption growth may be due to the fact that the variable is correlated with realized consumption growth - the true risk factor. To test this, we directly test whether \( b_2 = 0 \) and \( b_2 = b_3 = 0 \). A rejection of these hypotheses is evidence in support of our three-factor model, where expected consumption growth shocks are a priced risk factor. Panel B of Table 6 shows that this is indeed the case (rejections at the 10% level). The statistical evidence is not very strong, but on the other hand we are relying on proxies and the sample is fairly small.

We conclude that the linear three-factor model derived from our theoretical model outperforms the benchmark Consumption CAPM. We reject the null hypothesis that long-run risk is not important relative to the standard Consumption CAPM for the cross-section of stock returns.
Table 6
The Price of Long-Run Risk from Cross-Sectional Regressions

Table 6: This table reports the estimated prices of risk for the factors of the Consumption CAPM and the long-run risk production-based model developed in this paper (Prod.CAPM). Test assets are the 25 Fama-French portfolios sorted by size and book-to-market equity ratios. All variables are annual. There are 57 observations from 1949 - 2005. Estimation is by two-pass regression, where the standard errors are corrected for generated regressors (Shanken, 1992). P-values are reported for each variable, with the null hypothesis is that the estimate is zero. Numbers in bold indicate significance at the 10% level or more in a two-tailed t-test.

Panel A: Estimation of Factor Risk Prices

<table>
<thead>
<tr>
<th>Factor Risk Price ($\lambda$)</th>
<th>Cons.CAPM</th>
<th>Prod.CAPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Realized Cons. Growth ($\lambda_{\Delta c}$)</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>(p-value)</td>
<td>(0.24)</td>
<td>(0.97)</td>
</tr>
<tr>
<td>Shock to Exp. Cons. Growth ($\lambda_c$)</td>
<td>$&gt; 0$</td>
<td></td>
</tr>
<tr>
<td>(p-value)</td>
<td>(0.07)</td>
<td></td>
</tr>
<tr>
<td>Expected Cons. Growth ($\lambda_x$)</td>
<td>$&lt; 0$</td>
<td></td>
</tr>
<tr>
<td>(p-value)</td>
<td>(0.11)</td>
<td></td>
</tr>
<tr>
<td>$R^2_{adj}$</td>
<td>17.4%</td>
<td>46.7%</td>
</tr>
</tbody>
</table>

Panel B: Tests of Factor Significance: $m_{t+1} = a - b_1 \Delta c_{t+1} - b_2 e_{t+1} - b_3 x_t$

<table>
<thead>
<tr>
<th></th>
<th>Cons.CAPM</th>
<th>Prod.CAPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_1 = 0?$</td>
<td>Don't reject</td>
<td>Don't reject</td>
</tr>
<tr>
<td>(p - val)</td>
<td>(0.24)</td>
<td>(0.82)</td>
</tr>
<tr>
<td>$b_2 = 0?$</td>
<td>reject</td>
<td></td>
</tr>
<tr>
<td>(p - val)</td>
<td>(0.09)</td>
<td></td>
</tr>
<tr>
<td>$b_2 = b_3 = 0?$</td>
<td>reject</td>
<td></td>
</tr>
<tr>
<td>(p - val)</td>
<td>(0.06)</td>
<td></td>
</tr>
</tbody>
</table>
6 Conclusion

We analyze a standard production economy model where agents have Epstein-Zin preferences. We show that long-run risk arises endogenously as a consequence of consumption smoothing, even though log technology follows a random walk without time-varying expected growth rates. When the coefficient of relative risk aversion is greater than the reciprocal of the elasticity of intertemporal substitution, agents dislike negative shocks to future economic growth prospects and the time-varying expected consumption growth rate appears as a risk factor. We show that in this case the presence of long-run risk decreases the market price of risk if technology shocks are transitory, while it increases the market price of risk if technology shocks are permanent. We furthermore show that this effect is quantitatively important when we calibrate the model to match macroeconomic moments. The endogenous long-run risk process is quantitatively very close to processes that have been specified as exogenous in the literature. We conclude that long-run consumption risk is the natural assumption, given our theoretical models, for exogenous consumption growth processes in exchange economies. However, the impact this risk factor has on the total price of risk in the economy and the equity Sharpe ratio depends on both preferences and whether technology shocks are transitory or permanent.

The production economy model identifies the ratio of technology to consumption as a proxy for otherwise unobservable expected consumption growth. We test this link in the time-series of consumption growth and in the cross-section of stock returns. We find support for both tests. In particular, the production-based CAPM outperforms the standard CCAPM in a cross-sectional test. The parameter estimates obtained from the cross-sectional analysis are consistent with a model where technology shocks are permanent and agents have a preference for early resolution of uncertainty.

7 References


8 Appendix

8.1 Model Solution

The Return to Investment and the Firm’s Problem The firm maximizes firm value. Let $M_{t,t+1}$ denote the stochastic discount factor. The firm’s problem is then:

$$\max_{\{I_t,K_{t+1},W_t\}_{t=0}^T} E_0 \left[ \sum_{t=0}^{T} M_{0,t} \left\{ (Y_t - W_t H_t - I_t) - q_t \left( K_{t+1} - (1 - \delta) K_t - \phi \left( \frac{I_t}{K_t} \right) K_t \right) \right\} \right],$$

(28)

where $q_t$ denotes the shadow price of the capital accumulation constraint, equivalent to marginal $q$: The expected present value of one marginal unit of capital. Maximizing over labor we obtain $(1 - \alpha) Z_t K_t^\alpha H_t^{-\alpha} = W_t$ and $H_t = \left( (1 - \alpha) \frac{Z_t}{W_t} \right)^{\frac{1}{\alpha}} K_t$. The operating profit function of the firm follows as:

$$\Pi (K_t, Z_t; W_t) = Z_t K_t^\alpha \left[ \left( (1 - \alpha) \frac{Z_t}{W_t} \right)^{\frac{1}{\alpha}} K_t \right]^{1-\alpha} - W_t \left( (1 - \alpha) \frac{Z_t}{W_t} \right)^{\frac{1}{\alpha}} K_t$$

$$= K_t \left( (1 - \alpha)^{\frac{1}{\alpha}-1} Z_t^\frac{1}{\alpha} W_t^{1-\frac{1}{\alpha}} - (1 - \alpha)^{\frac{1}{\alpha}} Z_t^\frac{1}{\alpha} W_t^{1-\frac{1}{\alpha}} \right)$$

$$= K_t \left( \alpha (1 - \alpha)^{\frac{1}{\alpha}-1} Z_t^\frac{1}{\alpha} W_t^{1-\frac{1}{\alpha}} \right).$$

(29)
The operating profit function of the firm is thus linearly homogenous in capital. Substituting out equilibrium wages we obtain \( \Pi(K_t, Z_t; W_t) = \alpha Y_t \). We re-state the firm’s problem:

\[
\max_{(I_t, K_{t+1})} E_0 \left[ \sum_{t=0}^{\infty} M_{0,t} \left\{ \Pi(\cdot) - I_t - q_t \left( K_{t+1} - (1 - \delta) K_t - \phi \left( \frac{I_t}{K_t} \right) K_t \right) \right\} \right]. \tag{30}
\]

Each period in time the firm decides how much to invest, taking marginal \( q \) as given. The first order conditions with respect to \( I_t \) and \( K_{t+1} \) are immediate:

\[
0 = -1 + q_t \phi' \left( \frac{I_t}{K_t} \right), \tag{31}
\]

and

\[
0 = -q_t + E_t \left[ M_{t+1} \left\{ +q_{t+1} \left( (1 - \delta) - \phi' \left( \frac{I_{t+1}}{K_{t+1}} \right) \frac{I_{t+1}}{K_{t+1}} + \phi \left( \frac{I_{t+1}}{K_{t+1}} \right) \right) \right\} \right]. \tag{32}
\]

Substituting out \( q_t \) and \( q_{t+1} \) in (32) yields:

\[
\frac{1}{\phi' \left( \frac{I_t}{K_t} \right)} = E_t \left[ M_{t+1} \left\{ \Pi_K(\cdot) + \frac{(1 - \delta) - \phi' \left( \frac{I_{t+1}}{K_{t+1}} \right) \frac{I_{t+1}}{K_{t+1}} + \phi \left( \frac{I_{t+1}}{K_{t+1}} \right)}{\phi' \left( \frac{I_{t+1}}{K_{t+1}} \right)} \right\} \right], \tag{33}
\]

\[
1 = E_t \left[ M_{t+1} \left\{ \phi' \left( \frac{I_t}{K_t} \right) \left( \Pi_K(\cdot) + \frac{1 - \delta + \phi \left( \frac{I_{t+1}}{K_{t+1}} \right) - \frac{I_{t+1}}{K_{t+1}}}{\phi' \left( \frac{I_{t+1}}{K_{t+1}} \right)} \right) \right\} \right], \tag{34}
\]

\[
1 = E_t \left[ M_{t+1} R_{t+1} \right]. \tag{35}
\]

Equation (35) is the familiar law of one price, with the firm’s return to investment:

\[
R_{t+1} = \phi' \left( \frac{I_t}{K_t} \right) \left( \Pi_K(K_{t+1}, Z_{t+1}; W_{t+1}) + \frac{1 - \delta + \phi \left( \frac{I_{t+1}}{K_{t+1}} \right)}{\phi' \left( \frac{I_{t+1}}{K_{t+1}} \right)} - \frac{I_{t+1}}{K_{t+1}} \right). \tag{36}
\]

8.2 Risk and the Dynamic Behavior of Consumption

Epstein-Zin preferences have been used with increasing success in the asset pricing literature over the last years (e.g., Bansal and Yaron, 2004, Hansen, Heaton and Li, 2005, Malloy, Moskowitz and Vissing-Jorgensen, 2005, Yogo, 2006). This is both due to their recursive nature, which allows time-varying growth rates to increase the volatility of the stochastic discount factor through the return on the wealth portfolio, as well as the fact that these
preferences allow a convenient separation of the elasticity of intertemporal substitution from the coefficient of relative risk aversion.

Departing from time-separable power utility preferences with $\gamma = \frac{1}{\psi}$ means agents care about the temporal distribution of risk. In particular, Epstein and Zin (1989) show that $\gamma > \frac{1}{\psi}$ implies a preference for early resolution of uncertainty. This is a key assumption of our analysis, because it is precisely this departure from the classic preference structure that renders time-varying expected consumption growth rates induced by optimal consumption smoothing behavior a priced risk factor in the economy.

8.2.1 Early Resolution of Uncertainty and Aversion to Time-Varying Growth Rates

To gain some intuition for why a preference for early resolution of uncertainty implies aversion to time-varying growth rates, we revisit an example put forward in Duffie and Epstein (1992). Consider a world where each period of time consumption can be either high or low. Next, the consumer is given a choice between two consumption gambles, $A$ and $B$. Gamble $A$ entails eating $C_0 = \frac{1}{2}C^H + \frac{1}{2}C^L$ today, where $C^H$ is a high consumption level and $C^L$ is a low consumption level. Tomorrow you flip a fair coin. If the coin comes up heads, you will get $C^H$ each period forever. If the coin comes up tails, you will get $C^L$ each period forever. Gamble $B$ entails eating $C_0$ today, and then flip a fair coin each subsequent period $t$. If the coin comes up heads at time $t$, you get $C^H$ at time $t$, and if it comes out tails, you get $C^L$ at time $t$. Thus, in the first case uncertainty about future consumption is resolved early, while in the second case uncertainty is resolved gradually (late). If $\gamma = \frac{1}{\psi}$ (power utility), the consumer is indifferent with respect to the timing of the resolution of uncertainty and thus indifferent between the two gambles. However, an agent who prefers early resolution of uncertainty (i.e., she likes to plan), prefers gamble $A$.

We can now also phrase this discussion in terms of growth rates. From this perspective, gamble $A$ has constant expected consumption growth, while gamble $B$ has a mean-reverting process for expected consumption growth. Thus, a preference for early resolution of uncertainty translates into an aversion of time-varying expected consumption growth.

Another, more mechanical, way to see this is by directly looking at the stochastic discount factor. It is well known, e.g. Rubinstein (1976), that the stochastic discount factor, $M_{t+1}$, is the ratio of the representative agent’s marginal utility between today and tomorrow:
\[ M_{t+1} = \frac{U'(C_{t+1})}{U'(C_t)}. \]

Using a recursive argument, Epstein and Zin (1989) show that:

\[ \ln M_{t+1} \equiv m_{t+1} = \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1 - \theta) r_{a,t+1}, \quad (37) \]

where \( \Delta c_{t+1} \equiv \ln \frac{c_{t+1}}{c_t} \) and \( r_{a,t+1} \equiv \ln \frac{c_{t+1} + A_{t+1}}{A_t} \) is the return on the total wealth portfolio with \( A_t \) denoting total wealth at time \( t \).\(^{16}\) If \( \gamma = \frac{1}{\psi} \), \( \theta = \frac{1-\gamma}{1-1/\psi} = 1 \), and the stochastic discount factor collapses to the familiar power utility case. However, if the agent prefers early resolution of uncertainty, the return on the wealth portfolio appears as a risk factor. More time-variation in expected consumption growth (the expected "dividends" on the total wealth portfolio) induces higher volatility of asset returns, in turn resulting in a more volatile stochastic discount factor and thus a higher price of risk in the economy.\(^{17}\)

The effect on the equity premium can be understood by considering a log-linear approximation (see Campbell, 1999) of returns and the pricing kernel, yielding the following expressions for the risk-free rate and the equity premium:

\[ r_{f,t+1} \approx -\log \beta + \frac{1}{\psi} E_t [\Delta c_{t+1}] - \frac{\theta}{2\psi^2 \sigma_{t,c}^2} + \frac{(\theta - 1)}{2} \sigma_{t,r}^2, \quad (38) \]

\[ E_t [r_{t+1}^E] - r_{f,t+1} \approx \frac{\theta}{\psi} \sigma_{t,r} \sigma_{t,E} + (1 - \theta) \sigma_{t,r} \sigma_{t,E} - \frac{\sigma_{t,r}^2}{2}, \quad (39) \]

where \( E_t [\Delta c_{t+1}] \) is expected log consumption growth, \( \sigma_{t,c}, \sigma_{t,r}, \sigma_{t,E} \), are the conditional standard deviations of log consumption growth, the log return on the total wealth portfolio, and the log equity return, and \( \sigma_{t,r} \) \( \sigma_{t,E} \) are the conditional covariances of the log equity return with log consumption growth and the log return on the total wealth portfolio respectively. We can see how the level of the equity premium depends directly on the covariance of equity returns with returns on the wealth portfolio.

\[ \text{8.2.2 Technology and Risk Aversion} \]

Standard production technologies do not allow agents to hedge the technology shock. Agents must in the aggregate hold the claim to the firm’s dividends. Therefore, the only action available to agents at time \( t \) in terms of hedging the shock at time \( t + 1 \), is to increase

\(^{16}\)Note that our representative household’s total wealth portfolio is composed of the present value of future labor income in addition to the value of the firm.

\(^{17}\)This assumes that the correlation between the return on the wealth portfolio and consumption growth is non-negative, which it is for all parameter values we consider in this paper (and many more).
savings in order to increase wealth for time $t + 1$. The shock will still hit the agents at time $t + 1$ though, no matter what. Wealth levels may be higher if a bad realization of the technology shock hits the agents, but wealth is also higher if a good realization of the technology shock occurs. The difference between the agents’ utility for a good realization of the technology shock in period $t + 1$ relative to their utility for a bad realization of the shock is thus (almost) unaffected. However, it is this utility difference the agents care about in terms of their risk aversion. Now, because the agents’ utility function is concave, this is not quite true. A higher wealth level in both states of the world does decrease the difference between utility levels. Agents thus respond by building up what is referred to as "buffer-stock-savings". This is, however, a second-order effect. As a result, the dynamic behavior of consumption growth is largely unaffected by changing agents’ coefficient of relative risk aversion. The fundamental consumption risk in the economy remains therefore (almost) the same when we increase risk aversion ($\gamma$) while holding the $EIS$ ($\psi$) constant. Asset prices, of course, respond as usual to higher levels of risk aversion.

Table 7 confirms this result. We report Model 3 and Model 6 from Table 2 with a coefficient of relative risk aversion ($\gamma$) of 5, as well as versions of the models with a higher level of risk aversion ($\gamma = 25$).

8.2.3 A "Matching the Moments" Experiment

Higher capital adjustment costs ($CAC$) result in higher volatility of equity returns. However, this comes at a cost: The volatility of investment, and thereby the magnitude of long-run risk in the economy, decreases. In the previous sections we have constrained the $EIS$ to be less than or equal to 1.5 and the coefficient of relative risk aversion ($\gamma$) to be 5. These are levels that have been suggested as economically reasonable in the literature (e.g., Mehra and Prescott, 1985, on relative risk aversion and Campbell, 1999, on $EIS$). In the following experiment, we let these parameters be free to see how well the "unrestricted" production economy model can match financial moments. The parameters are set to jointly match consumption growth volatility, the relative volatility of consumption and output growth, the Sharpe ratio of equity returns, and the average level of the risk-free rate as well as possible.\textsuperscript{18} Table 8 displays the results. Models 8 and 12 are repeated from Table 3 as the "restricted" production economy counterparts. Model 18 (transitory shocks) fits all the moments except the risk-free rate, which is too high. However, the level of relative risk

\textsuperscript{18}Model 12 from Table 3 already fits all of these moments. Thus, the additional criterion is to maximize the equity premium.
Table 7

The Effect of Risk Aversion on Macroeconomic Time Series

Table 7: This table reports relevant macroeconomic moments and consumption dynamics for models with either transitory ($\varphi = 0.90$) or permanent technology shocks and different levels of the coefficient of relative risk aversion. The elasticity of intertemporal substitution ($\psi$) is 1.5 across all models. We re-calibrate the discount factor ($\beta$) for each model so as to jointly match the values for (C/Y), (I/Y), (D/Y), with each model. Capital adjustment costs ($\xi$) are 30 in order to match the relative volatility of consumption to output with Model 6. Model 3 and Model 15 share the same parameter values apart from the coefficient of relative risk aversion ($\gamma$). The same is true for Models 6 and 16. We estimate the following process for the consumption dynamics:

$$c_{t+1} = \mu + x_t + \sigma \eta_{t+1}, \quad x_{t+1} = \rho x_t + \sigma e_{t+1}.$$ 

The standard deviation of variable $X$ denotes $\sigma[X]$. We use annual U.S. data from 1929 to 1998 from the Bureau of Economic Analysis. The sample is the same as in Bansal and Yaron (2004). Under Panel B we report the calibration of the exogenous consumption process Bansal and Yaron use. All values reported in the table are quarterly.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Model 3</th>
<th>Model 16</th>
<th>Model 6</th>
<th>Model 17</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transitory Shocks</td>
<td>Permanent Shocks</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_{t+1} = \mu t + \varphi z_t + \sigma \xi_{t+1}$</td>
<td>$z_{t+1} = \mu + z_t + \sigma \xi_{t+1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma = 5$</td>
<td>$\gamma = 25$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Panel A: Macroeconomic Moments (Quarterly)

<table>
<thead>
<tr>
<th>U.S. Data</th>
<th>1929-1998</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma[\Delta y]$ (%)</td>
<td>2.62</td>
</tr>
<tr>
<td>$\sigma[\Delta c]/\sigma[\Delta y]$</td>
<td>0.52</td>
</tr>
<tr>
<td>$\sigma[\Delta i]/\sigma[\Delta y]$</td>
<td>3.32</td>
</tr>
<tr>
<td>$E[C/Y]$</td>
<td>0.79</td>
</tr>
</tbody>
</table>

### Panel B: Consumption Dynamics: $\Delta c_{t+1} = \mu + x_t + \sigma \eta_{t+1}, \quad x_{t+1} = \rho x_t + \sigma e_{t+1}$. 

<table>
<thead>
<tr>
<th>Bansal, Yaron Calibration</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma[\Delta c]$ (%)</td>
</tr>
<tr>
<td>$\sigma_\eta$ (%)</td>
</tr>
<tr>
<td>$\sigma[x]$ (%)</td>
</tr>
<tr>
<td>$\rho$</td>
</tr>
<tr>
<td>$\sigma_e$ (%)</td>
</tr>
</tbody>
</table>
aversion, $\gamma$, is 100. Note that this level of risk aversion is substantially more than what would be required in a standard production economy model with power utility. The reason is that long-run risk decreases the market price of risk in the model with Epstein-Zin preferences, as explained earlier. The power utility model on the other hand can not at all simultaneously fit the relative volatility of consumption and output, the Sharpe ratio, and the risk-free rate (Weil, 1989). The model with permanent shocks (Model 19) does fit all the moments. The (unreasonably) high EIS of 15 allows a higher level of CAC and equity volatility increases from 0.57% to 3.67%.

We conclude that none of the models can generate realistic equity return volatilities while matching key macroeconomic moments at the same time, even with unrealistically high levels of risk aversion or elasticity of intertemporal substitution.

8.3 Accuracy of the Approximation of the Endogenous Consumption Process

In section 4.2 we propose the following approximation for the dynamics of the endogenous process for consumption:

\[
\Delta c_{t+1} = \mu + x_t + \sigma_\eta \eta_{t+1}, \tag{40}
\]

\[
x_{t+1} = \rho x_t + \sigma_e e_{t+1}, \tag{41}
\]

\[
\sigma_{\eta,e} = \text{corr} \left( \eta_{t+1}, e_{t+1} \right). \tag{42}
\]

Here $\Delta c_{t+1}$ is log realized consumption growth, $x_t$ is the time-varying component of expected consumption growth, and $\eta_t, e_t$ are zero mean, unit variance, and normally distributed disturbance terms with correlation $\sigma_{\eta,e}$. This functional form for log consumption growth is identical to the one assumed by Bansal and Yaron (2004) as driving process of their exchange economy model. Our results therefore provide a theoretical justification for their particular exogenous consumption growth process assumption. To evaluate whether the above specified process is a good approximation of the true consumption growth dynamics we first estimate the process from simulated data for a whole range of different model calibrations both with random walk- as well as with AR(1) technology processes. Then we compare the autocorrelation function obtained directly from the simulated data to the one implied by the above specified process which we have imposed on the data.

For the random walk technology the autocorrelation functions are virtually indistin-
Table 8

Financial Moments: High Values for $\gamma$ and $\psi$

Table 8: This table reports relevant financial moments and consumption dynamics for models with either transitory ($\varphi = 0.90$) or permanent technology shocks. We allow (counterfactually) higher values for the coefficient of relative risk aversion ($\gamma$) and the elasticity of intertemporal substitution ($\psi$) in order to demonstrate the degree to which the models can so fit the data. We estimate the following process for the consumption dynamics: $\Delta c_{t+1} = \mu + \varphi z_t + \sigma \varepsilon_{t+1}$, $x_{t+1} = \rho x_t + \sigma e_{t+1}$. $\Delta x = \log(X_t) - \log(X_{t-1})$, and $\sigma[X]$ denotes the standard deviation of variable $X$. The data are taken from Bansal and Yaron (2004) who use annual U.S. data from 1929 to 1998, except real, per capita GDP growth which we obtain from the Bureau of Economic Analysis. All values reported in the table are annual.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Model 8</th>
<th>Model 18</th>
<th>Model 12</th>
<th>Model 19</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transitory Shocks</td>
<td>Permanent Shocks</td>
<td>Transitory Shocks</td>
<td>Permanent Shocks</td>
<td></td>
</tr>
<tr>
<td>$z_{t+1} = \mu t + \varphi z_t + \sigma \varepsilon_{t+1}$</td>
<td>$z_{t+1} = \mu + z_t + \sigma \varepsilon_{t+1}$</td>
<td>$\beta = 0.998$</td>
<td>$\beta = 0.99995$</td>
<td></td>
</tr>
<tr>
<td>$\gamma = 5, \psi = 0.50$</td>
<td>$\gamma = 100, \psi = 0.50$</td>
<td>$\gamma = 5, \psi = 1.50$</td>
<td>$\gamma = 4.8, \psi = 15$</td>
<td></td>
</tr>
<tr>
<td>$\xi = 22$</td>
<td>$\xi = 3$</td>
<td>$\xi = 22$</td>
<td>$\xi = 3$</td>
<td></td>
</tr>
</tbody>
</table>

Panel A: The Price of Risk and Consumption Dynamics (Annual)

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Model 8</th>
<th>Model 18</th>
<th>Model 12</th>
<th>Model 19</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma[\Delta c]/\sigma[\Delta y]$</td>
<td>0.52</td>
<td>0.26</td>
<td>0.52</td>
<td>0.52</td>
</tr>
<tr>
<td>$\sigma[\Delta c]$ (%)</td>
<td>2.72</td>
<td>3.22</td>
<td>2.72</td>
<td>2.72</td>
</tr>
<tr>
<td>$\sigma [x]$ (%)</td>
<td>$n/a$</td>
<td>0.14</td>
<td>0.21</td>
<td>0.33</td>
</tr>
<tr>
<td>$\sigma [M]/E [M]$</td>
<td>$n/a$</td>
<td>0.08</td>
<td>0.35</td>
<td>0.34</td>
</tr>
<tr>
<td>$SR [\varepsilon^E]$</td>
<td>0.33</td>
<td>0.07</td>
<td>0.33</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Panel B: Financial Moments (Annual)

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Model 8</th>
<th>Model 18</th>
<th>Model 12</th>
<th>Model 19</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E [r_f]$ (%)</td>
<td>0.86</td>
<td>3.68</td>
<td>1.51</td>
<td>0.86</td>
</tr>
<tr>
<td>$\sigma [r_f]$ (%)</td>
<td>0.97</td>
<td>0.55</td>
<td>0.81</td>
<td>0.43</td>
</tr>
<tr>
<td>$E [r_E - r_f]$ (%)</td>
<td>6.33</td>
<td>0.12</td>
<td>1.31</td>
<td>0.19</td>
</tr>
<tr>
<td>$\sigma [r_E - r_f]$ (%)</td>
<td>19.42</td>
<td>1.77</td>
<td>3.94</td>
<td>0.57</td>
</tr>
</tbody>
</table>
Figure 4: **Autocorrelation Functions Consumption Growth** Comparison of the autocorrelation function obtained directly from simulated data of Model 3 and Model 6 to the autocorrelation function implied by the postulated process for expected consumption growth which we have estimated from the same simulated data of Model 3 and Model 6 respectively.
guishable in all cases we have examined. Figure 4 shows this for Model 6. For the AR(1) technology the approximation turns out to get worse the lower the persistence of the driving process. Figure 4 shows the autocorrelation functions for Model 3. A look at Figure 1 makes clear why the above specified approximation for the dynamics of the endogenous process for consumption is worse for the case where technology shocks are transitory, because the impulse-response of consumption to technology shocks is "hump-shaped". We therefore conclude that our postulated process is a good representation of the endogenous consumption growth dynamics for models with highly persistent technology shocks.

8.4 Numerical Solution

8.4.1 Solution Algorithm

We solve the following model:

\[
V(K_t, Z_t) = \max_{C_{t+1}, K_{t+1}} \left\{ \left[ (1 - \beta) C_t^{1-\gamma} + \beta \left( E_t \left[ V(K_{t+1}, Z_{t+1})^{1-\gamma} \right] \right)^{1-\gamma} \right]^{1-\phi} \right\},
\]

\[K_{t+1} = (1 - \delta) K_t + \phi \left( \frac{I_t}{K_t} \right) K_t,
\]

\[I_t = Y_t - C_t,
\]

\[Y_t = Z_t^{(1-\alpha)} K_t^{\alpha},
\]

\[\ln Z_{t+1} = \phi \ln Z_t + \varepsilon_{t+1},
\]

\[\varepsilon_t \sim N(\mu, \sigma_z).
\]

We focus in this appendix on the case where \( \phi = 1 \). Since then the process for productivity is non-stationary, we need to normalize the economy by \( Z_t \), in order to be able to numerically solve the model. To be precise, we let \( \hat{K}_t = \frac{K_t}{Z_t}, \hat{C}_t = \frac{C_t}{Z_t}, \hat{I}_t = \frac{I_t}{Z_t}, \) and substitute. In the

\(^{19}\)We assume the disturbance terms \( \eta \) and \( \varepsilon \) to be i.i.d. normally distributed. The shocks we obtain when we estimate our postulated process for consumption growth from simulated data turn out to be very close to normal. They display mild heteroscedasticity.

\(^{20}\)This conclusion relies on the assumption that the consumption process is covariance-stationary, which it is since the production function is constant returns to scale and preferences are homothetic. The autocorrelation function is then one of the fundamental time series representations. See, e.g., Hamilton (1994).
so transformed model all variables are stationary. The only state variable of the normalized model is $\hat{K}$.\footnote{Note that $Z$ is \textit{not} a state variable of the normalized model. This is due to the fact that we assume the autoregressive coefficient of the process for productivity $\ln \hat{Z}_{t+1} = \rho \ln Z_t + \varepsilon_{t+1}$ to be unity: $\rho = 1$. As a consequence, $\Delta Z$ is serially uncorrelated.} We can work directly on the appropriately normalized set of equations and then re-normalize after having solved the model.\footnote{In the paper we also report results for models where $\rho < 1$. In this case we work directly on the above non-normalized set of equations. The state variables are then $K$ and $Z$. The solution algorithm is identical to the case where $\rho = 1$.}

The value function is given by:

$$
\hat{V}(\hat{K}_t) = \max_{\hat{C}_t, \hat{K}_{t+1}} \left\{ \left[ (1 - \beta) \hat{C}_t^{1-\gamma} + \beta \left( E_t \left[ (\varepsilon_{t+1})^{1-\gamma} \left( \hat{V}(\hat{K}_{t+1}) \right)^{1-\gamma} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\theta}{1-\gamma}} \right\}.
$$

We parameterize the value function with a 5th order Chebyshev orthogonal polynomial over a $6 \times 1$ Chebyshev grid for the state variable $\hat{K}$:

$$
\Psi^A(\hat{K}) = \hat{V}(\hat{K}).
$$

We use the value function iteration algorithm. At each grid point for the state $\hat{K}$, given a polynomial for the value function $\Psi^A(\hat{K})$, we use a numerical optimizer to find the policy $(\hat{C}^*)$ that maximizes the value function:

$$
\hat{K}_{t+1}^* e^{\varepsilon_{t+1}} = \hat{Y}_t - \hat{C}_t^* + (1 - \delta) \hat{K}_t,
$$

$$
\hat{V}^*(\hat{K}_t) = \left[ (1 - \beta) \left( \hat{C}_t^* \right)^{\frac{1-\gamma}{\gamma}} + \beta \left( E_t \left[ (\varepsilon_{t+1})^{1-\gamma} \left( \Psi^A(\hat{K}_{t+1}^*) \right)^{1-\gamma} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\theta}{1-\gamma}},
$$

where Gauss-Hermite quadrature with 5 nodes is used to approximate the expectations operator. We use a regression of $\hat{V}^*$ on $\hat{K}$ in order to update the coefficients of the polynomial for the value function and so obtain $\Psi^A_{t+1}(\hat{K})$.

### 8.5 The Linear Factor Model

The log stochastic discount factor is:

$$
m_{t+1} = \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1 - \theta) r_{a,t+1},
$$

\footnote{Note that $Z$ is \textit{not} a state variable of the normalized model. This is due to the fact that we assume the autoregressive coefficient of the process for productivity $\ln \hat{Z}_{t+1} = \rho \ln Z_t + \varepsilon_{t+1}$ to be unity: $\rho = 1$. As a consequence, $\Delta Z$ is serially uncorrelated.}
where $\theta = \frac{1 - \gamma}{1 - \psi}$. The process for consumption growth is:

\[
\begin{align*}
\Delta c_{t+1} &= \mu + x_t + \sigma_w \eta_{t+1}, \quad (54) \\
x_{t+1} &= \rho x_t + \sigma_x e_{t+1}, \quad (55) \\
\sigma_{\eta,e} &= \text{corr} \left( \varepsilon_{t+1}, \eta_{t+1} \right). \quad (56)
\end{align*}
\]

Linearizing the wealth-consumption ratio around its steady state, we obtain (see Campbell, 1999, for a detailed derivation):

\[
r_{a,t+1} \approx \kappa_0 + \kappa_1 p c_{t+1} - p c_t + \Delta c_{t+1}, \quad (57)
\]

where $p c_t$ is the log wealth-consumption ratio, $\kappa_1 = \frac{\exp(p c_t)}{1 + \exp(p c_t)} \approx 0.96$, and $p c_t$ is the steady state log wealth-consumption ratio. Assuming log aggregate consumption growth $\Delta c_{t+1}$ to follow (54), Bansal and Yaron (2004) show that the log wealth-consumption ratio can be written as:

\[
p c_{t+1} \approx A_0 + A_1 x_{t+1}, \quad (58)
\]

where

\[
A_1 = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho}. \quad (59)
\]

Since $0 < \rho < 1$, $0 < \kappa_1 \rho < 1$. Thus, $A_1 > (\leq) 0$ if $\psi > (\leq) 1$. We substitute for $r_{a,t+1}$ in the log stochastic discount factor (53):

\[
m_{t+1} \approx \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1 - \theta) \kappa_0 \\
- (1 - \theta) \kappa_1 p c_{t+1} + (1 - \theta) p c_t - (1 - \theta) \Delta c_{t+1} \\
\approx \theta \ln \beta - (1 - \theta) \kappa_0 - \left(1 - \theta + \frac{\theta}{\psi}\right) \Delta c_{t+1} \\
- (1 - \theta) \left[ \kappa_1 A_0 + \kappa_1 A_1 x_{t+1} - A_0 - A_1 x_t \right] \\
= \theta \ln \beta - (1 - \theta) \kappa_0 - \left(1 - \theta + \frac{\theta}{\psi}\right) \Delta c_{t+1} - (1 - \theta) A_0 \kappa_1 + (1 - \theta) A_0 \\
- (1 - \theta) \left[ \kappa_1 A_1 \left( \rho x_t + \sigma_x e_{t+1} \right) - A_1 x_t \right] \\
= \theta \ln \beta - (1 - \theta) \kappa_0 - (1 - \theta) A_0 \kappa_1 + (1 - \theta) A_0 - \left(1 - \theta + \frac{\theta}{\psi}\right) \Delta c_{t+1} \\
- (1 - \theta) A_1 \kappa_1 \rho x_t - (1 - \theta) A_1 \kappa_1 \sigma_x e_{t+1} + (1 - \theta) A_1 x_t. \quad (60)
\]
Let:

\[ \alpha = \theta \ln \beta - (1 - \theta) \kappa_0 - (1 - \theta) A_0 \kappa_1 + (1 - \theta) A_0. \]  

(61)

Then:

\[ m_{t+1} \approx \alpha - \gamma \Delta c_{t+1} - (1 - \theta) A_1 \kappa_1 \rho x_t - (1 - \theta) A_1 \kappa_1 \sigma_e e_{t+1} + (1 - \theta) A_1 x_t \]
\[ = \alpha - \gamma \Delta c_{t+1} + (1 - \theta) A_1 (1 - \kappa_1 \rho) x_t - (1 - \theta) A_1 \kappa_1 \sigma_e e_{t+1} \]
\[ = \alpha - \gamma (\mu + x_t + \sigma_\eta \eta_{t+1}) + (1 - \theta) A_1 (1 - \kappa_1 \rho) x_t - (1 - \theta) A_1 \kappa_1 \sigma_e e_{t+1}. \]  

(62)

We derive:

\[ m_{t+1} - E_t [m_{t+1}] \approx -\gamma \sigma_\eta \eta_{t+1} - (1 - \theta) A_1 \kappa_1 \sigma_e e_{t+1} \]
\[ = -\gamma \sigma_\eta \eta_{t+1} - \left( \frac{\gamma - \frac{1}{\psi}}{1 - \kappa_1 \rho} \right) \kappa_1 \sigma_e e_{t+1} \]
\[ = -\gamma \sigma_\eta \eta_{t+1} + \left( \frac{1}{\psi} - \gamma \right) A_2 \sigma_e e_{t+1} \]
\[ = -\gamma \left[ \sigma_\eta \eta_{t+1} + A_2 \sigma_e e_{t+1} \right] + \frac{1}{\psi} A_2 \sigma_e e_{t+1}, \]  

(63)  

(64)  

(65)

(66)

where \( A_2 = \frac{\kappa_1}{1 - \kappa_1 \rho} > 0. \) We write equation (62) as:

\[ m_{t+1} \approx a - b_1 \Delta c_{t+1} - b_2 e_{t+1} - b_3 x_t, \]  

(67)

where \( b_1 = \gamma, \) \( b_2 = (\theta - 1) A_1 (1 - \kappa_1 \rho), \) \( b_2 = (1 - \theta) A_1 \kappa_1 \sigma_e \) are all greater than zero. Note that \( (1 - \theta) A_1 = \frac{\gamma - \frac{1}{\psi}}{1 - \kappa_1 \rho}. \) Thus, if \( \gamma > \frac{1}{\psi}, \) then \( (1 - \theta) A_1 > 0. \) By applying a standard log-linear first-order approximation (see, e.g., Yogo, 2006, for a similar application), the (not log) stochastic discount factor can be written as:

\[ \frac{M_t}{E[M_t]} \approx 1 + m_t - E[m_t]. \]  

(68)

This in turn implies a linear unconditional factor model (see Cochrane, 2001):

\[ E[R_{i,t+1} - R_{0,t+1}] = b_1 \text{Cov}(\Delta c_{t+1}, R_{i,t+1} - R_{0,t+1}) + b_2 \text{Cov}(e_{t+1}, R_{i,t+1} - R_{0,t+1}) \]
\[ + b_3 \text{Cov}(x_t, R_{i,t+1} - R_{0,t+1}), \]  

(69)
where $R_{i,t}$ denotes the time $t$ gross return on asset $i$, and $R_{0,t}$ denotes the time $t$ gross return on a reference asset (the risk-free rate). Recall that $b_1, b_2 > 0, b_3 < 0$. The price of risk of both realized consumption growth ($\Delta c_{t+1}$) and the shock to expected consumption growth ($e_{t+1}$) is thus positive, while the exposure to the time $t$ level of consumption growth ($x_t$) carries a negative price of risk. Why is that? Note that the corresponding conditional factor model contains only two risk factors:

$$E_t [R_{i,t+1} - R_{0,t+1}] = b_1 \text{Cov}_t (\Delta c_{t+1}, R_{i,t+1} - R_{0,t+1}) + b_2 \text{Cov}_t (e_{t+1}, R_{i,t+1} - R_{0,t+1}) .$$

(70)

The factor $x_t$ thus stems from "conditioning down" the model, because we use the unconditional covariance of returns with total consumption growth and not with the innovation to consumption growth. This unconditional covariance would overstate the covariance of returns with innovations to consumption growth if $\text{Cov} (x_t, R_{i,t+1} - R_{0,t+1}) > 0$ and induce too high expected excess returns. The term $b_3 \text{Cov} (x_t, R_{i,t+1} - R_{0,t+1})$ corrects for this effect.