We study the pricing and capacity allocation problem of a service provider who serves two distinct customer classes. Customers within each class are inherently heterogeneous in their willingness to pay for service, but their utilities are also affected by the presence of other customers in the system. Specifically, customer utilities depend on how many customers are in the system at the time of service as well as who these other customers are. If the service provider can price discriminate between customer classes, pricing out a class, i.e., operating an exclusive system, can sometimes be optimal and that depends only on classes’ perceptions about each other. If the provider must charge a single price, an exclusive system is even more likely. We extend our analysis to a service provider who can prevent class interaction by allocating separate capacity segments to the two customer classes. Under price discrimination, allocating capacity is optimal if our measure of net appreciation between classes is negative. However, under a single-price policy, allocating capacity can be optimal even if this measure is positive. In fact, we show that the nature of asymmetry eventually determines the optimal strategy.

**Key words:** customer mix, crowding, pricing, capacity allocation

1. **Introduction**

In many service systems, service is simultaneously delivered to many customers who share the same physical environment. For example, members of a gym workout in the same space and share the equipment with other members, passengers on a cruise ship share the common areas on the ship, and customers of a nightclub enjoy the dancefloor with all the other customers. In such facilities, an individual customer’s perception of the service quality is highly influenced by the composition of the customer clientele. For example, some female gym members do not enjoy sharing the same facility with males, and in nightclubs and bars, males typically have strong preferences for other customers being female (Skinner et al. 2005, Kubacki et al. 2007). Other service settings where customer satisfaction is influenced by the others’ characteristics (such as age, economic class, social status, intellectual capabilities, etc.) include social clubs, health clubs, schools (Buchanan 1965,
Demand management is always a challenge but it is particularly important for such service establishments where each customer’s satisfaction depends on who the fellow customers are. The service provider who is facing this challenge has two powerful tools: pricing and capacity allocation. In its more extreme form, capacity allocation can even mean choosing to serve only certain segments of the population. This last alternative, i.e., restricting access to certain customer segments, may seem like a radical solution to managing the customer mix, but in practice it is more prevalent than expected. Such a restriction could be direct or a result of a “forced” self-selection. Gyms and health clubs employ direct restriction when they choose to become women–only establishments or allocate certain times of the week for the exclusive use of family with kids. On the other hand, some firms design the service experience so as to appeal to a particular segment and let the customers self-select. This is the idea behind cruise ships offering various themes (e.g., church cruises, cruises for singles with or without age restrictions, Star Trek fans, etc.) and nightclubs catering to different types of clientele on different floors of the venue or at different nights of the week by modifying the music choices and decoration. If such capacity allocation or restriction options are not available, or as a complementary tool, firms also use pricing as a means to manage their capacity and composition of their clientele, and maximize their profits. For example, nightclubs use various pricing promotions (e.g., “ladies’ nights”) to attract the “right” mix of customers.

Such practices are prevalent but that does not mean that they are devoid of controversy. “Ladies’ nights” have long been criticized by some as being discriminatory against men and this led to a number of lawsuits being filed over the years. Rulings have been mixed, with the practice being declared as legal in some states and illegal in others (Rank 2011). Recently some gyms have been the center of attention because of their policy to restrict service to women, or design special areas and/or times for the exclusive use of women. In 2007, a man filed a complaint with the Nevada Equal Rights Commission arguing that the Las Vegas Athletic Club’s practice of charging a reduced sign-up fee and providing a special workout area for women was illegal under state law (Friess 2007). The commission decided that the price differences were illegal, but providing private workout areas was not (Friess 2008). Interestingly, Nevada passed a law making gender-based pricing legal if made for promotional purposes in 2011 (Schoenmann 2011). A more recent controversy was caused by Fitness USA, which abruptly decided to make two of its locations in Michigan women-only. Angered customers, mostly male, cited various reasons for their protests; some seem to have been caused by precipitous transition. One female customer stated that her monthly membership fee went from $19 to $24 due to this switch (Komar 2013). Apparently, the company preferred to offer...
its services exclusively to females and charge them a higher price, even if that meant angering some of its customers. In general, it is important to note that even though women-only health clubs draw ire occasionally and some argue that they are or should be illegal, they are nevertheless highly popular and common in and outside the U.S. Moreover, and despite the controversies and the legal concerns, the revenues associated with the leisure industry, where customer mix effects are prominent, are quite high. In the UK, it generates over £200bn of revenue every year, provides 2.6m jobs and represents 9% of the workforce (Wyman 2012). Similarly, in the US, the health club industry has annual revenues of $27bn (IBISWorld 2014b) and the nightlife industry has annual revenues of $24bn (IBISWorld 2014a). All these figures point to the importance of investigating the optimal pricing and capacity allocation strategies in these contexts.

In the establishments described above, the two fundamental questions the service provider needs to answer are: given the available capacity, what is the “optimal” customer mix and how should that mix be achieved? The objective of this paper is to provide insights into these two questions, which are inextricably linked. The optimal mix could be so that the system is an exclusive one where service is offered to one segment of the population or an inclusive one where customers from different segments interact. Alternatively, the provider may also choose to allocate capacity for the exclusive use of each segment. Another interesting dimension is whether or not and how firm’s pricing policy affects such decisions. Our analysis sheds light on these questions, helps identify conditions that would lead firms to choose one strategy over the other, and explains some of the existing practices we observe in the service industry.

The main challenge in investigating these questions is that no prior work can serve as the foundation for our modeling effort. Despite the fact that the operations management literature is rich in articles that deal with pricing, demand management, and capacity control in the context of service operations, the focus is not on the service process itself. Specifically, the “service” experience in these articles is typically not influenced by the characteristics of the others with whom they share the service experience (or service is simply not a shared experience), whereas delays in access to service is the important dimension of the problem. As a consequence of that, most papers consider queueing-based formulations. In contrast, for the service settings we are interested in, capturing the delivery of the service process (specifically, who the other customers are and how many of them there are)—as opposed to delays in access to service—is far more important. Thus, one of the main contributions of this paper is the development of a novel stylized formulation that permits detailed analysis of pricing and capacity allocation decisions for such settings.

Our model assumes that the service provider serves two classes of customers. Customers of one class have stochastically larger intrinsic valuations for the service they will receive. Each customer is assumed to know the distribution of service valuations for both customer classes and uses this
information along with the price to decide whether or not to purchase service. We focus first on the pricing decision alone and assume that the provider does not have the option to allocate different capacity segments to different customer classes, but she can deny service to one of the two classes altogether. We consider two different settings; in Section 4.1, the firm has the flexibility to charge different prices to different classes, and in Section 4.2, the firm has to charge the same price to all customers. When price discrimination is allowed, the firm might choose to exclude a particular class from service only due to the classes’ perceptions of each other, i.e. the customer mix effects. Additionally, increasing the capacity might increase utilization. This surprising phenomenon is observed when customers are symmetric in their inherent willingness to pay for service and the customer mix effects are mild but disappears as the asymmetry increases. When the firm is forced to choose a single price, a strong asymmetry in the feelings of the two classes about each other urges the provider to restrict access to a single class in order to be profitable. Interestingly, this is not true when there is mutual dislike. This suggests that attempts to achieve price “fairness” by disallowing price discrimination might lead the service provider to deny service to one class.

In Sections 4.3 and 4.4 of the paper, assuming that the firm can allocate capacity to different customer classes, we study the optimal allocation and pricing policy. In Section 5, we compare the different strategies to shed light into the design of such a service system and we find that if the firm can price discriminate, whether or not the firm chooses to allocate capacity depends purely on classes’ perceptions about each other, not on any potential willingness–to–pay asymmetry between classes. However, this choice is more complicated if the firm has to charge the same price to both classes. In most cases, we again find that a firm that cannot price discriminate is more likely to prefer capacity allocation compared to a firm that can charge different prices; however, this is not always true if customer classes are asymmetric in their inherent willingness to pay for service. In Section 6, we gain further insights into the customer mix effects using numerical examples. We also discuss the robustness of our results through a sensitivity analysis.

2. Literature Review
As we discussed in Section 1, prior work in the operations management literature has mostly investigated questions related to pricing and capacity control in the context of service establishments where queueing prior to service is a very important aspect of the service experience. Thus, this body of work typically considers models that capture congestion effects and where customers are possibly delay-sensitive (e.g., Naor 1969, Mendelson 1985, Mendelson and Whang 1990, Afêche 2013, Afêche and Pavlin 2013) and/or queue lengths provide signals regarding the service quality (e.g., see Debo and Veeraraghavan 2009, Veeraraghavan and Debo 2009, 2011). This is unlike our formulation, which does not incorporate customer waiting in any explicit manner. For the service
settings we are interested in, capturing the service process during delivery, not the delays in access to service, is far more relevant. Specifically, we focus on the consumption of a service good where class heterogeneity and the total number of customers has an impact on the customers’ utility. To the best of our knowledge, the effects of this customer-to-customer interactions and their influence on the firms’ pricing and capacity allocation decisions, has not been analytically studied before.

One paper that is relatively closer to our work is Johari and Kumar (2010), which considers positive-only network effects together with congestion effects. The authors are mainly motivated by online services and these two effects are formulated in a way that is more general than our approach in that the effects not only depend on the number of active users in the system but also on the load these users generate. However, unlike the case in our model, Johari and Kumar assume that all customers have the same utilities under the same set of conditions. Thus, their model ignores possible asymmetry in how customers from different segments feel about each other. Furthermore, their focus is completely different from ours. The authors are not interested in pricing and capacity allocation decisions for a profit–maximizing firm, but rather focus on the optimal number of users from two different perspectives: the user-preferred and the manager-preferred. The gap between the two optima is discussed along with its implications.

In the economics literature, there are some articles related to our paper. A significant portion of these articles belong to a stream of work on “club theory,” which originated from the seminal papers by Tiebout (1956) and Buchanan (1965). (For an extensive review of this literature, see Cornes and Sandler 1996 and Sandler and Tschirhart 1997.) However, this literature typically investigates questions that are completely different from ours. Specifically, except for a few papers (Hearne 1988, Basu 1989), the traditional club theory has not focused on pricing and/or capacity allocation considerations of a profit–maximizing firm. Moreover, again except for a few papers (e.g., Basu 1989, Brueckner and Lee 1989, Scotchmer 1997, Becker and Murphy 2000), the club theory literature has typically assumed that customers are homogeneous and their utilities do not depend on the characteristics of the other individuals in the facility.

The five papers/chapters we cite above as exceptions (Hearne 1988, Basu 1989, Brueckner and Lee 1989, Scotchmer 1997, Becker and Murphy 2000) may need further elaboration to clarify in what sense they are different from this paper. Hearne (1988) considers the profit–maximization problem of a monopolistic club and shows that a two–part tariff mechanism (membership and per usage fees) will lead to a Pareto–optimal solution. Apart from the focus, the paper is different from ours in that the customers are assumed to be homogeneous. Basu (1989) is generally interested in developing a theory of association and studies a number of different formulations motivated by different applications. In one of his models, in the schools context, he considers four types of students in the population: clever and rich, clever and poor, mediocre and rich, and mediocre and poor.
Rich students are willing to pay more than poor students and (rich or poor) students’ willingness to pay depends on what fraction of the school population is clever. This work differs from ours mainly in that Basu (1989) is purely interested in the question of whether the schools should be allowed to charge different prices to different types of students, not providing insights into a profit-maximizing service provider’s optimal pricing and capacity allocation decisions. Brueckner and Lee (1989) are also motivated by schools. They assume that there are two groups in the population and the utility of the customers from one of the groups depends on the proportion of customers from the other group. The paper characterizes the Pareto-efficient club configurations and carries out an equilibrium analysis for a competition model. Scotchmer (1997) considers a formulation in which the utility of each customer type depends on the number of customers from each type. She defines a new notion of approximate competitive equilibrium and shows that there exists such an equilibrium when the economy is sufficiently large. Note that neither Brueckner and Lee (1989) nor Scotchmer (1997) develop insights into the optimal pricing and capacity allocation decisions from an individual club’s perspective. The model of Chapter 5 in Becker and Murphy (2000) seems to be the most relevant to our work because it also assumes that the utility of a customer depends on the ratio of customers from one class. Despite this similarity, however, the objective is significantly different from ours. In particular, they assume that prices are determined through a competitive bidding process and there exists no service provider who sets prices to maximize profit or revenue.

Outside the club theory literature, another stream of articles within the economics literature deals with systems whose customers experience positive network effects. Armstrong (2006) and Rochet and Tirole (2003) study two-sided markets where the two groups of agents interact via a, not necessarily physical, platform and the focus is on pricing mechanisms to attract the right mix of agents from both groups and achieve a “good” balance. Since the focus is not restricted to physical platforms, there is no consideration of capacity allocation nor crowding effects and the main attention is driven to a competitive environment and mechanisms to gain a larger market share. There is also some literature that refers to the “network effect” as the effect that other users in the network have on the utility of an individual user. For example, see Oren and Smith (1981), and more recently Candogan et al. (2012). These articles ignore the possibility that network effects across different groups within the population could be different. To the best of our knowledge, the only exception to this is Katz and Spiegel (1996) that uses a similar demand formulation but with no capacity considerations. There is also a large body of work that focuses on congestion effects leaving out positive network externalities. For examples of such work, we refer the reader to MacKie-Mason and Varian (1995), Wang and Schulzrinne (2006), and references therein.

Finally, there are many articles in the marketing literature that investigate customer-to-customer interactions (CCI) in services (see Nicholls 2010 for an extensive review). A number of articles
empirically study CCI in various service environments including nightclubs (Skinner et al. 2005, Kubacki et al. 2007), professional conferences (Gruen et al. 2007), adventure sports (Thakor et al. 2008), beauty salons (Moore et al. 2005), cruise ships (Huang and Hsu 2010), and organized tours (Wu 2007), and find that customers can have strong preferences regarding who they share their service experience with. Moreover, some articles discuss the importance of the management of CCI in the service industry in general and point to various strategies the providers might employ. Among these, Martin (1996) and Grove and Fisk (1997) discuss operational issues including the effective use of capacity, which we also address in this paper. In particular, Martin (1996) investigates customers’ perceptions of and reactions to the others’ behavior. He suggests capacity allocation either through physical separation or time allocation for the use of different segments to improve the service experience of customers who might not enjoy each other’s company. This is a practice widely used and we also investigate it. In the same spirit, Grove and Fisk (1997) are interested in the effect of presence and behavior of others in customers’ service satisfaction. They find that it might not be ideal to operate systems at their maximum capacity and call for more research into identifying the optimal capacity for systems that serve many customers simultaneously. One of the contributions of this paper is establishing conditions under which the system’s capacity is fully utilized or underutilized.

3. Model
We consider a service system associated with a leisure facility with capacity $K > 0$ which serves two distinct customer classes, each one with the same finite size $\Lambda > 0$. We later consider different class sizes in a numerical study. Class membership of a customer is observable to the service provider and to all the other customers. Customers enjoy the leisure facility and their utilities consist of three different components and depend on $\lambda_1$ and $\lambda_2$, the number of customers in the system belonging to class 1 and 2, respectively. To facilitate game theoretic treatment we treat customers as non-atomic (infinitesimal) and therefore, $\lambda_1$ and $\lambda_2$ as continuous parameters. In the absence of other customers, the service value of class–1 customers are uniformly distributed on the line segment $[0,1]$. Likewise, the service value of class–2 customers are uniformly distributed on the line segment $[a,1+a]$, $a \geq 0$. Thus, on average, class–2 customers have the same or larger inherent willingness to pay for service than class–1 customers. In the presence of other customers, however, there are two components that may affect customer utility. Customers of a particular class might like or dislike sharing the same service environment with the other class. Moreover, their satisfaction can be dependent on the overall crowd size. In mathematical terms, the gross utilities $U_1$ and $U_2$ of a customer $x$ in class 1 and 2, respectively, are given by

$$U_1(x, \lambda_1, \lambda_2) = x + b_1 \lambda_2 / (\lambda_1 + \lambda_2) + c ((\lambda_1 + \lambda_2) / K), \quad 0 \leq x \leq 1,$$

$$U_2(x, \lambda_2, \lambda_1) = x + b_2 \lambda_1 / (\lambda_1 + \lambda_2) + c ((\lambda_1 + \lambda_2) / K), \quad a \leq x \leq 1 + a. \quad (1)$$
The term $b_i \lambda_{3-i} / (\lambda_1 + \lambda_2)$, $i = 1, 2$ in (1) captures the customer-mix effect on class--$i$ customer utilities. We assume that customers of each class are homogenous in their perception of the customer mix and this is represented by the parameter $b_i$. If $b_i > 0$ ($b_i < 0$), customers of class $i$ prefer a customer mix with more (fewer) class-$(3-i)$ customers. We also define $b = b_1 + b_2$ as the net "appreciation" between the two customer classes and will be useful in presenting our results.

Customers’ experience might also be affected by the crowding level—the ratio of the total number of customers ($\lambda_1 + \lambda_2$) to the system’s capacity ($K$). Depending on the leisure activity, an undercrowded system or/and an overcrowded system might not be desirable for an enjoyable experience, which in turn reduces customer utility. The continuously differentiable function $c: [0, 1] \rightarrow \mathbb{R}$ in (1) captures these effects on customer utilities. We assume that $c''(\cdot) < 0$, thereby guaranteeing a uniquely optimal crowding level for an arbitrary customer. To avoid an empty system in equilibrium, we also assume that $c(0) > -1$. It is important to note that we impose no further restrictions on $c(\cdot)$; it can take positive or negative values, it can be monotone or unimodal. In fact, there are some service experiences where the overall crowding level in the system may not influence customers’ utilities, i.e. $c \equiv 0$, or service experiences where the customers’ utilities are affected only for certain crowding levels. In both cases, our results still hold. However, we assume that whatever the crowding effects are, they are symmetric across classes.

We consider a game in which the leisure facility chooses the prices $(p_1, p_2)$ simultaneously and commits to them. The customers arrive to the service facility, observe the price $p_i$, if from the class $i$, and decide whether to join the system or not. Assuming that the joining strategy is of a threshold type, a customer $x_i$ from class $i$ arriving to the service facility will join the system if $x_i \geq x_i^*$, otherwise, he will not. This implies that

$$\lambda_1^* = \Lambda(1 - x_1^*) \text{ and } \lambda_2^* = \Lambda(a + 1 - x_2^*)$$

due to the uniform distribution of the service value. From equation (1), there exists a marginal customer from each class whose utility will satisfy

$$x_1^* + b_1 \frac{a + 1 - x_2^*}{a + 2 - x_1^* - x_2^*} + c \left( \frac{\Lambda(a + 2 - x_1^* - x_2^*)}{K} \right) = p_1,$$

$$x_2^* + b_2 \frac{1 - x_1^*}{a + 2 - x_1^* - x_2^*} + c \left( \frac{\Lambda(a + 2 - x_1^* - x_2^*)}{K} \right) = p_2.$$

The solution of this system, $(x_1^*, x_2^*)$ will denote the Nash equilibrium (NE) of the game. Since there is a unique mapping between $(x_1^*, x_2^*)$ and $(\lambda_1, \lambda_2)$, the NE can be equivalently expressed in terms of $(\lambda_1, \lambda_2)$ and the equilibrium prices will be derived as follows

$$p_1(\lambda_1, \lambda_2) = 1 - \lambda_1 / \Lambda + b_1 \lambda_2 / (\lambda_1 + \lambda_2) + c ((\lambda_1 + \lambda_2) / K),$$

$$p_2(\lambda_1, \lambda_2) = 1 + a - \lambda_2 / \Lambda + b_2 \lambda_1 / (\lambda_1 + \lambda_2) + c ((\lambda_1 + \lambda_2) / K).$$
The structure of the solution is provided separately for the different cases in the next sections. The uniqueness of the optimal solution confirms that the joining strategy is of a threshold type. Because customer utilities depend on $\lambda_1$ and $\lambda_2$, which are equilibrium quantities, a potential customer must construct beliefs about the equilibrium values $\lambda_1$ and $\lambda_2$ when deciding whether or not to join the system. In turn, these beliefs must be confirmed in equilibrium, that is, customers should act rationally with respect to information and be able to correctly predict the equilibrium values $\lambda_1$ and $\lambda_2$ as a result. As in all definitions of equilibrium, customers choices and beliefs are determined simultaneously.

Before moving on to the analysis of the service provider’s optimization problem in the next section, we briefly comment on the case in which classes are identical and customer-mix effects do not exist or are ignored, i.e., $a = 0$ and $b_1 = b_2 = 0$. In that case, it is easy to show that the service provider 1) always prefers to have both classes in the system to sustain higher prices; 2) charges both classes the same price even when price discrimination is allowed. Therefore, if classes are identical and the customer mix does not affect customer utilities, neither capacity allocation nor price discrimination are of any value to a service provider. As we demonstrate in this paper, asymmetry in the willingness to pay for service and/or customer-mix effects make both price discrimination and capacity allocation effective tools to the service providers, and explain to a great extent what is observed in practice.

To help with the exposition in the rest of the paper, we introduce the following terminology; we call a system exclusive, if no interaction between the two classes is allowed and inclusive otherwise. Exclusivity can be a result of restricting access to a single class or allocating capacity for the exclusive use of each class. We call a system full if its crowding level is equal to one; we call a system not full if its crowding level is strictly less than one. Also, we refer to the case $a = 0$ as symmetric and to the case $a > 0$ as asymmetric with classes described as being symmetric and asymmetric respectively. Notice in equation (1) that the two customer classes are possibly different in two dimensions: their feelings about each other and their (inherent) willingness to pay for service and therefore, our definition of symmetry is with a slight abuse of the terminology$^1$.

4. Optimal pricing and capacity allocation decisions

We start our analysis in Section 4.1 with a leisure facility where the two classes share the whole capacity and the service provider is allowed to charge them differently. We call this scenario price discrimination without capacity allocation (CS-DP). We continue in Section 4.2 with the more restrictive pricing policy, where the provider must charge the same price to all customers and we

$^1$ Classes are truly symmetric only if $a = 0$ and $b_1 = b_2$. 
will call this scenario single price without capacity allocation (CS-SP). In both CS-DP and CS-SP, however, the provider can choose to restrict access to members of one class only, i.e., run an exclusive system.

If the service provider is better off running an exclusive system, the provider might, in fact, increase revenue by allocating separate capacity segments for the exclusive use of each customer class. The service provider might be able to divert customers to the “right” location depending on their class identities or she can design the service and the service environment for different segments in a way that will induce customers to self-select. In a leisure facility, the manager can have two designated areas for the two classes. In a nightclub, this usually happens by hosting theme nights on different days of the week so as to appeal to customers with particular tastes and interests. Nightclubs with adequate space might also provide a private area for members who are willing to pay a premium so as not to socialize with the rest of the clientele. The manager can also allocate each floor of a building to a different class of customers. In Section 4.3, we study the service provider’s problem under the assumption that she exercises her option to allocate capacity to each customer class and price discriminate and we call this scenario price discrimination with capacity allocation (CA-DP). We then restrict the problem to the single price case in Section 4.4 and we call this scenario single price with capacity allocation (CA-SP). We use (P1), (P2), (P3), and (P4) to represent the mathematical formulations of the optimization problems that correspond to CS-DP, CS-SP, CA-DP and CA-SP, respectively.

4.1. Price discrimination without capacity allocation

We start our analysis with a leisure facility, a nightclub for instance, where the two classes, the male and the female customers, share the whole capacity and the service provider is allowed to charge them differently. In that setting, typically male customers are willing to pay more, not for the service per se but because they are considered to gain more from the interaction with female customers, than female customers do (Armstrong 2006). The service provider’s objective is to charge prices so as to maximize the total profit. Hence, an individually rational provider who can charge a different price to each class maximizes revenue by solving the following problem:

$$\max_{\lambda_1, \lambda_2} R(\lambda_1, \lambda_2) = \lambda_1 p_1(\lambda_1, \lambda_2) + \lambda_2 p_2(\lambda_2, \lambda_1)$$

s.t. $$\lambda_1 + \lambda_2 \leq K, \ 0 \leq \lambda_1 \leq \Lambda, \ 0 \leq \lambda_2 \leq \Lambda.$$ (P1)

We first provide the following results, which establish some basic properties of the optimal solution ($\lambda^*_1, \lambda^*_2$) to problem (P1).

**Lemma 1.** (i) If $a = 0$, then $\lambda^*_1 \lambda^*_2 > 0$ if and only if $\lambda^*_1 = \lambda^*_2$.

(ii) If $a = 0$, a feasible solution to (P1) at which $\lambda_1 = \lambda > 0, \lambda_2 = 0$, is revenue-equivalent to a feasible solution to (P1) at which $\lambda_1 = 0, \lambda_2 = \lambda > 0$. 
If $a > 0$, then $\lambda^*_2 > \lambda^*_1$.

The properties described in Lemma 1 may seem rather intuitive but what is striking is that the customer mix effects captured with parameters $b_1$ and $b_2$ play no role. According to the lemma, even if class-1 customers are very fond of class-2 customers but the latter despise the former, the provider will admit the same number of customers from each class if $a = 0$, and will admit more customers from class 2 if $a > 0$. This may seem to suggest that when it comes to the customer mix in equilibrium, the customer-mix effects are irrelevant. However, the customer mix effects implicitly play a role when the provider has to determine if she will operate an exclusive or an inclusive leisure facility ($\lambda^*_1 \lambda^*_2 > 0$ or $\lambda^*_1 \lambda^*_2 = 0$), as we will see in Proposition 1 below. Nevertheless, it is true that if it is optimal for the provider to admit both classes, most customers will be from the class that has the higher willingness to pay for service regardless of any asymmetry in how classes feel about being around each other. This result is due to the provider’s ability to internalize any asymmetry in the customer-mix effects by charging different prices. For example, the male customers of a nightclub might end up paying a much higher price than the female customers; in fact, the price differential will be so large that the same number of customers from both classes will eventually choose to join the system.

The next proposition characterizes the general structure of the NE, i.e., the structure of the optimal solution to (P1).

**PROPOSITION 1.** If customers from different classes are allowed to share the same space and the service provider can price discriminate, the optimal solution to the revenue maximization problem has the following properties:

(i) There exists threshold $b^*(K)$ such that $\lambda^*_1 = 0$, $\lambda^*_2 > 0$ if $b \leq b^*(K)$, and $\lambda^*_2 \geq \lambda^*_1 > 0$ if $b > b^*(K)$.

(ii) If $K \leq \min\{\Lambda(1 + a + c(1))/2, 2(1 + c(1))\Lambda/3\}$, then $\lambda^*_1 + \lambda^*_2 = K$.

(iii) If $K$ is sufficiently large, then $\lambda^*_1 + \lambda^*_2 < K$.

(iv) If $b$ is sufficiently positive so that $\lambda^*_1 > 0 \forall K$, or if $b$ is sufficiently negative so that $\lambda^*_1 = 0 \forall K$, then there exists $K^+(b)$ such that $\lambda^*_1 + \lambda^*_2 = K$ if $K \leq K^+(b)$, and $\lambda^*_1 + \lambda^*_2 < K$ if $K > K^+(b)$.

Proposition 1 characterizes the basic structure of the NE and provides insights into the two key decisions the service provider needs to make. First, she needs to decide whether to admit customers from both classes (inclusive system) or to restrict access to the customers with higher willingness-to-pay (exclusive system). Second, she needs to decide whether the existing capacity should be fully utilized or intentionally kept underutilized at profit-maximizing prices. Figure 1 illustrates the different system types that arise in equilibrium if classes are symmetric (a) or asymmetric (b). In what follows, we first discuss the most important insights in the case of symmetric classes and then
we highlight the differences that arise if classes are asymmetric. When following this discussion, the reader would likely find it helpful to refer to the graphs in Figure 1.

In symmetric classes, Proposition 1-(i) states that although the system capacity is a factor in deciding whether or not the system should be inclusive or exclusive, when it comes to customer-mix effects, the individual terms \( b_1 \) and \( b_2 \) are irrelevant given the net appreciation term \( b \). More specifically, if the net appreciation between classes is sufficiently negative, the provider is better off leaving one class out of the system. Although it is possible that one class likes the other (e.g., \( b_1 > 0 \)), if the feelings of the other class are opposite and much more intense (i.e., \( b_2 \ll -b_1 \)), then an exclusive system helps prevent customer-mix effects from hurting revenues. For example, in a cruise ship, couples with children would likely not be bothered by couples without kids, but couples without kids might be unhappy about the presence of children running around. Similarly, some female customers of gyms and health clubs are not willing to share the same workout space with male customers. In either case, if this disutility is strong, it is best to run an exclusive system. This might be the motivation behind Fitness USA decision to go women-only. It is also the case that smaller capacities are more likely to be filled up by top-paying customers only and be profitable.

Parts (ii), (iii), and (iv) of Proposition 1 characterize how the choice regarding the number of customers in the system should be made to fill in the capacity. Not surprisingly, if the capacity is sufficiently small, there are enough customers who would be willing to pay a high price and the system will be fully utilized regardless of an exclusive or inclusive system. On the other hand, if capacity is very large, running a full system is suboptimal as it would necessitate charging unjustifiably low prices or would be outright impossible.
Part (iv) of Proposition 1 strengthens these structural properties further. When customer-mix effects are so powerful that a system is either always inclusive or always exclusive regardless of its capacity, then progressively larger capacities can only imply transitions from full to not full systems. However, if the customer-mix effects are relatively weak, which is possibly the most common scenario in a leisure facility, then we have some interesting and unexpected changes in the preference for exclusivity and crowding level (Figure 1(a)). We use a numerical example to illustrate this. Consider a small absolute value of the net appreciation effect, $b = -1.1$, and $\Lambda = 100$, $a = 0$. If $K = 45$, the system is in the regime of part (ii) of Proposition 1, i.e., a full exclusive system is optimal and the corresponding revenue is $R(0, 45) = 24.75$. On the other hand, the highest revenue an inclusive system could yield is $R(22.5, 22.5) = 22.5$. In this case, the limited capacity does not allow the provider to adequately counter the negative customer-mix effects by admitting more customers from both classes. Suppose that capacity increases to $K = 51$. Now, the most profitable system is still exclusive but not full any more, with 50 customers and revenue $R(0, 50) = 25$, whereas the highest revenue an inclusive system could yield is $R(25.5, 25.5) = 23.97$. In this case, again, capacity is not sufficient to result in enough revenue for an inclusive system to be optimal. Finally, suppose that capacity increases even further to $K = 60$. The optimal system now is full and inclusive, with 30 customers from each class and revenue $R(30, 30) = 25.5$. On the other hand, the highest revenue an exclusive system could yield remains at $R(0, 50) = 25$. At this capacity level, the provider can admit enough customers from both classes to make up for the revenue she loses due to negative customer-mix effects. It is the negative customer interaction effects that hurt revenues of inclusive systems, thus making it difficult to make a general statement about the effect of capacity changes based on intuition alone. In the absence of such effects, admitting customers from both classes would raise the average price customers pay compared to an exclusive system with the same number of customers.

In asymmetric classes, the asymmetry in the willingness to pay for service does not change the overall structure of the equilibrium substantially (Figure 1(b)). However, there are two noteworthy differences. First, the net appreciation between classes needs to be higher for inclusivity to be the optimal choice because the provider can simply find more customers in class–2 than in class–1 to pay a good price for service. Second, when the two classes are strongly asymmetric, as in the case of Figure 1(b), an increase in the system capacity can never result in the optimal crowding level changing from “not full” to “full.” This is in contrast to the symmetric and weakly asymmetric cases (Figure 1(a)), where a capacity increase can switch the optimal policy from “exclusive, not full” to “inclusive, full.” This difference is due to the fact that class-1 customers’ low willingness-to-pay combined with sufficiently strong negative customer effects between the two classes does
not justify admitting class-1 customers in the more asymmetric cases. Thus, the system remains
exclusive as capacity increases and operating a full system does not become a better alternative.

We conclude this section with a comparison of the prices that classes pay when they coexist. The
revenue achieved by the service provider depends on the overall asymmetry of the classes \((b, a)\) that
determines the proportion of the customers that will join the facility \((\lambda_1, \lambda_2)\). The extra flexibility
that she possesses, she uses it by charging prices that reflect the classes’ feelings; higher \(b_i\) implies
higher \(p_i\). Equations (2) and (3) imply that
\[
p^*_2 - p^*_1 = a + (\lambda^*_1 - \lambda^*_2)/\Lambda + (b_2\lambda^*_1 - b_1\lambda^*_2)/(\lambda^*_1 + \lambda^*_2).
\]
Wherefore, if classes are symmetric and the provider runs an inclusive system, the class that likes
dislikes the other the most (the least) pays a higher price for service and in particular, \(p^*_2 - p^*_1 =
(b_2 - b_1)/2\). This might explain why “ladies’ night” is a common promotional event at nightclubs or
why some colleges offer reduced tuition to students of high caliber. “Ladies” are offered discounts to
compensate for their relatively stronger disutility (or weaker utility) of having “gentlemen” around.
With asymmetric classes \((a > 0)\), the price comparison is not straightforward. In this case, \(\lambda^*_1 < \lambda^*_2\)
and class–2 customers might end up paying less than class–1 customers, although they can afford
a higher price for service. The reason is that if class–1 customers value the presence of class 2 much
more than class–2 customers value them in return \((b_1 >> b_2)\), the former will end up paying more
than the latter although they are not as wealthy on average. This result partially explains why
famous and wealthy individuals enjoy a free ride at certain social events; the strong desire of less
wealthy and less famous people to be around them might give rise to this phenomenon.

4.2. Single price without capacity allocation

As discussed in Section 1, price discrimination is a sensitive issue and can be illegal, or not ethical,
when it is based on a demographic factor. Whether or not it is implemented depends on a com-
bination of factors including what the law says about the practice, whether the law is enforced,
customers’ attitude, and the provider’s ability to manage customer perceptions. When the man-
ger is constrained by these factors to charging a single price, she has to either charge the optimal
unique price or she may offer the service to only one of the two classes (and thereby make the
single price constraint meaningless).

Using (2) and (3), the constraint \(p_1 = p_2\) implies
\[
[b_1/(\lambda_1 + \lambda_2) + 1/\Lambda]\lambda_2 = [b_2/(\lambda_1 + \lambda_2) + 1/\Lambda]\lambda_1 + a.
\]
Because \(a \geq 0\), a NE in which the provider charges a single price and \(\lambda^*_1\lambda^*_2 > 0\) is possible only if
\(b_1 < 0\) and \(b_2 < -a\), or if \(b_2 > -K/\Lambda\) and \(b_1 > a - K/\Lambda\). Hence, without proof, the following lemma.

**Lemma 2.** The service provider can charge a single price in a NE in which \(\lambda^*_1\lambda^*_2 > 0\) only if
\(b_1 < 0\) and \(b_2 < -a\), or if \(b_2 > -K/\Lambda\) and \(b_1 > a - K/\Lambda\).
The necessary conditions of Lemma 2 essentially say that there is a limit to how differently the two customer classes can feel about each other and still allow a profitable single-price policy with both classes being admitted. Interestingly, if the dislike between customer classes is mutual, this is not sufficient for the provider to deny service to one of the classes. In that case, there are always customers who are willing to pay the asking price and bear with the customers from the other class due to the inherent heterogeneity within customer classes. The intensity of the customer feelings determines the ratio of the two classes in the facility and as a result, when there is strong asymmetry in the two classes’ mutual appreciation, it is not profitable to maintain an inclusive facility using a single price. Although Lemma 2 identifies conditions under which an inclusive system with single price might be profitable, the provider might be better off running an exclusive system (Figure 2).

We name the optimization problem the service provider needs to solve (P2). To solve (P2), the service provider first solves the following problem (P2′), which enforces the single-price constraint and ignores the possibility that the service can be limited to only one class. Problem (P2′) is essentially problem (P1) with the addition of the single-price constraint (4).

\[
\begin{align*}
\max_{\lambda_1, \lambda_2} & \quad R(\lambda_1, \lambda_2) = \lambda_1 p_1 (\lambda_1, \lambda_2) + \lambda_2 p_2 (\lambda_2, \lambda_1) \\
\text{s.t.} & \quad \lambda_1 + \lambda_2 \leq K \\
& \quad [b_1/(\lambda_1 + \lambda_2) + 1/\Lambda] \lambda_2 = [b_2/(\lambda_1 + \lambda_2) + 1/\Lambda] \lambda_1 + a \\
& \quad 0 \leq \lambda_1 \leq \Lambda, 0 \leq \lambda_2 \leq \Lambda.
\end{align*}
\]

(P2′)

The solution to (P2) is then obtained by comparing the optimal solution to (P2′) with the optimal solution under which the service is restricted to class-2 customers. (There is no need to consider the case where service is restricted to class-1 customers because such a solution is guaranteed to not lead to higher revenue. Restriction to either class leads to the same revenue only if \(a = 0\).)

We first provide the following results, which establish some basic properties of the optimal solution \((\lambda^*_1, \lambda^*_2)\) to problem (P2).

**Lemma 3.**

(i) If \(a = 0\) and \(b_i \geq b_{3-i}\), then either \(\lambda^*_i \lambda^*_{3-i} = 0\) or \(\lambda^*_i \geq \lambda^*_{3-i}\) \(i = 1, 2\).

(ii) If \(a = 0\), a feasible solution to (P2) at which \(\lambda_1 = \lambda > 0\), \(\lambda_2 = 0\), is revenue-equivalent to a feasible solution to (P2) at which \(\lambda_1 = 0\), \(\lambda_2 = \lambda > 0\).

According to Lemma 3, if classes are symmetric, the provider either admits only one customer class, or she runs an inclusive system with more customers from the class that likes (dislikes) the other more (less). To understand the last result, which is a departure from Lemma 1-(i), it is helpful to first recall that classes are not truly symmetric unless \(a = 0\) and \(b_1 = b_2 = 0\). If \(a = 0\) but \(b_1 \neq b_2\), the single-price constraint does not permit a customer mix with an equal number of customers from both classes. For example, if \(b_1 > b_2\) and the provider charges a single price, there will be more class-1 than class-2 customers who are willing to pay that price and the optimal customer mix will have more customers from class-1.
Figure 2  Structure of the optimal policy under single-price policy without capacity allocation when $\Lambda = 100$, $b_2 = 0.$

The next proposition characterizes the overall structure of the NE in the case of single price, i.e., the structure of the optimal solution to (P2). (We slightly abuse notation by using the same symbols, $b^*(K)$ and $K^*(b)$, in both propositions, although they correspond to different values in general.)

**Proposition 2.** If customers from both classes are allowed to share the same space and the service provider cannot price discriminate, the optimal solution has the following properties, where we let $\Delta b \equiv |b_1 - b_2|.$

(i) There exists a threshold $b^*(K)$ such that $\lambda_1^* = 0$ if $b \leq b^*(K)$, and $\lambda_1^* > 0$ if $b > b^*(K)$. Furthermore, if $a = 0$ and there exists a net appreciation value $\bar{b}$ such that $b_1 + b_2 = \bar{b}$ and $\lambda_1^* \lambda_2^* > 0$ if $\Delta \bar{b} = 0$, then there exists threshold $\Delta b^*(K) > 0$ such that $\lambda_1^* \lambda_2^* > 0$ if $\Delta b \leq \Delta b^*(K)$, and $\lambda_1^* \lambda_2^* = 0$ if $\Delta b > \Delta b^*(K)$.

(ii) If $K \leq \min\{\Lambda(1 + a + c(1))/2, \Lambda(1 + c(1))\}$, then $\lambda_1^* + \lambda_2^* = K$.

(iii) If $K$ is sufficiently large, then $\lambda_1^* + \lambda_2^* < K$.

(iv) If $b$ is sufficiently positive so that $\lambda_1^* > 0 \forall K$, or if $b$ is sufficiently negative so that $\lambda_1^* = 0 \forall K$, then there exists $K^*(b)$ such that $\lambda_1^* + \lambda_2^* = K$ if $K \leq K^*(b)$, and $\lambda_1^* + \lambda_2^* < K$ if $K > K^*(b)$.

A quick read of Proposition 2 reveals that each of its statements corresponds to an analogous statement in Proposition 1 which is also evident by comparing Figures 1 and 2. There is, however, one important difference. If the provider cannot price discriminate, both the net appreciation term $b (\equiv b_1 + b_2)$ and the individual terms $b_1$, $b_2$ matter. In fact, the second part of Proposition 2-(i) states that, for a given net appreciation term, unless the two individual terms $b_1$ and $b_2$ are sufficiently
close to each other, service will be restricted to one class. In other words, a single-price policy leads
to an exclusive system when this asymmetry is sufficiently large unlike in the price discrimination
case where the asymmetric customer-mix effects are absorbed by the differential pricing (Figures
1 and 2). The revenue is tightly constrained by the single-price condition. As we explained in the
discussion of Lemma 2, this condition critically depends on how different terms \( b_1 \) and \( b_2 \) are from
each other. Thus, when the provider charges a single price, not only the net appreciation term,
but also the individual terms \( b_1 \) and \( b_2 \) are important. In other words, the customer mix effects on
revenue, which are symmetric across classes under price discrimination, become asymmetric under
the single-price clause. This result practically implies that when regulators attempt to achieve
“price fairness” by disallowing price discrimination, they might inadvertently be forcing the service
provider to exclude an entire class of customers from service if that is practically feasible. Although
there is no evidence to conclude that this is the reason why gyms like Fitness USA, which we
discussed in Section 1, convert at least some of their locations to women-only establishments, they
are very likely to be affected by the same underlying dynamics. By restricting access to females,
these gyms not only make their service environment more appealing to women and increase their
willingness to pay for the overall service experience, but also bypass any possible restriction (legal
or otherwise) to charge the same price to both men and women. It is also interesting that, due
to this phenomenon, the optimal price may have a non-monotonic relationship with the capacity.
Specifically, one might expect that the optimal price would decrease with an increase in system
capacity but as it turns out, a larger capacity might mean the optimality of an inclusive system
with a higher price.

4.3. Price discrimination with capacity allocation

Capacity allocation with or without price discrimination is a prevalent practice. For example, theme
cruises typically occupy part of a cruise ship while the rest of the ship is filled with passengers on a
regular tour, or there could be multiple theme cruises simultaneously on the same ship. Similarly,
some gyms, health clubs, and public swimming pools allocate their capacity to different customer
classes through space separation or time allocation.

If the service provider can allocate capacity, she needs to decide the capacity to allocate to each
class as well as the optimal number of customers to admit. In the subsequent analysis, \((1 - x)K\)
denotes the fraction of capacity allocated to class–1 customers and \(xK\) denotes the fraction of
capacity allocated to class–2 customers. The equilibrium prices are modified as follows

\[
p_1(\lambda_1) = 1 - \frac{\lambda_1}{\Lambda} + c\left(\frac{\lambda_1}{((1 - x)K)}\right), \tag{5}
\]

\[
p_2(\lambda_2) = 1 - \frac{\lambda_2}{\Lambda} + a + c\left(\frac{\lambda_2}{(xK)}\right). \tag{6}
\]
Note here that the customer mix effects disappear since the two classes do not coexist. As previously, we study first the price discrimination policy (CA-DP) and in Section 4.4, we focus on the single-price policy (CA-SP).

Given the choices of capacity allocation and price discrimination, the service provider is faced with the following revenue maximization problem:

\[
\max_{\lambda_1, \lambda_2, x} \ R(\lambda_1, \lambda_2, x) = \lambda_1 p_1(\lambda_1) + \lambda_2 p_2(\lambda_2)
\]

\[
\text{s.t. } 0 \leq \lambda_1 \leq (1 - x)K
\]

\[
0 \leq \lambda_2 \leq xK
\]

\[
0 \leq x \leq 1.
\]

We first provide the following results, which establish the uniqueness of the optimal solution to problem (P3), as well as some important properties of \(x^*\), the optimal fraction of capacity allocated to class 2.

**Lemma 4.**

(i) There exists a unique optimal solution to (P3).

(ii) If \(\lambda_1^* \lambda_2^* > 0\), crowding levels are the same in both capacity segments, i.e., \(\lambda_1^*/[(1 - x^*)K] = \lambda_2^*/(x^*K)\).

(iii) The optimal allocation fraction for class 2, \(x^*\), equals \(\lambda_2^*/(\lambda_1^* + \lambda_2^*)\).

(iv) If \(a = 0\), \(x^* = 1/2\). In addition, \(x^*\) is increasing in \(a\).

Lemma 4 is a key result for the remainder of our analysis. The fact that the classes are identical with respect to their sensitivity towards crowding and that the crowding disutility function \(c\) is (strictly) concave explains the identical crowding levels in both segments. Furthermore, crowding levels are inextricably linked to each other because the two capacity segments share the same total capacity. As a result, there is a unique capacity allocation that makes crowding levels in the two segments equal to each other. In the absence of customer mix effect, the capacity will be equally split when the two classes are symmetric but, as expected, when they are not symmetric, more capacity will be allocated to the class that values the service more.

### 4.4. Single price with capacity allocation

In this section, we describe the optimization problem of the service provider when capacity allocation is an option but prices need to be the same for both classes. First, note that the single-price constraint is relevant only when \(0 < x < 1\). In that case, using equations (5) and (6) and enforcing \(p_1 = p_2\) yields the following condition

\[
\lambda_2/\Lambda - c(\lambda_2/(xK)) = \lambda_1/\Lambda - c(\lambda_1/((1 - x)K)) + a.
\]

We use (P4) to refer to the optimization problem the service provider needs to solve to determine the optimal allocation and the optimal demand levels. As in the case of capacity sharing with
single-price restriction, the single-price constraint disappears when $x = 0$ or $x = 1$, and the problem is solved in two stages. First, the service provider solves the following optimization problem:

$$\max_{\lambda_1, \lambda_2, x} R(\lambda_1, \lambda_2, x) = \lambda_1 p_1(\lambda_1) + \lambda_2 p_2(\lambda_2)$$

s.t.

$$0 \leq \lambda_1 \leq (1 - x)K$$
$$0 \leq \lambda_2 \leq xK$$
$$0 < x < 1$$
$$\lambda_2 / \Lambda - c(\lambda_2 / (xK)) = \lambda_1 / \Lambda - c(\lambda_1 / [(1 - x)K]) + a.$$

The solution to (P4) is then obtained by comparing the optimal solution to (P4') with the optimal solution under which the whole capacity is reserved for class-2 customers (equivalently class-1 customers only if $a = 0$) and this solution is unique (following a proof that is similar to the proof of Lemma 4 and thus is omitted). In the following section, we use the optimization problem (P4) to prove a number of results on how the policies compare with each other with respect to their optimal revenues.

5. Policy Comparison

In this section, we focus on the most important aspect of the service provider’s decision; the optimal policy to adopt depending on the attributes of the customer base. We compare the revenues under the four different scenarios and provide analytical results that are valuable for the design of such a service system.

We start with the case where the manager has the flexibility to charge different prices to the two customer classes and we establish a useful link between the optimal solution to problem (P1) and the optimal solution to problem (P3) in the next corollary.

Corollary 1. If the service provider can price discriminate and $b = 0$, the optimal revenue and customer mix are the same with or without capacity allocation.

Corollary 1 essentially says that if the net appreciation term is zero, the ability to allocate capacity does not change anything: the provider makes the same revenue with or without capacity allocation, and the resulting customer mix is the same. What is striking in this result is that significant asymmetry in how the two classes feel about each other (e.g., $b_1 >> 0$ and $b_2 << 0$) does not necessarily imply that the service provider would benefit from separating the two classes. If these asymmetric customer-mix effects are roughly the same in absolute value, then there is not much to gain from separation. Corollary 1 might leave one with the impression that prices with and without capacity allocation are the same. That is not true in general. Unless $b_1 = b_2 = 0$, a simple pairwise comparison of equations (2)–(3) and (5)–(6) reveals that the provider charges different prices when she allocates capacity and when she does not. For example, if $b_1 > 0, b_2 < 0, b_1 + b_2 = 0$, class-1 customers pay a lower price when the provider allocates capacity than when she does not.
because they lose the benefit of interacting with class-2 customers that they like. The opposite is true for class-2 customers. This price difference does not affect the net customer utility, which is the same in both settings, but points to an important implication of an operational decision: depending on whether or not the service provider uses capacity allocation, customers from both classes can end up enjoying different values from the service and paying significantly different prices without affecting the service provider’s revenue. We illustrate this in detail in Table 1 later.

In general, when \( b \neq 0 \), the service provider has to choose between capacity allocation and sharing. The next theorem provides sufficient conditions for the optimality of each strategy.\(^2\)

**Theorem 1.** If the service provider can allocate capacity and can price discriminate, the capacity allocation decision is as follows:

(i) If \( b \leq 0 \), it is optimal to allocate capacity.

(ii) If \( b \geq 0 \), it is optimal to not allocate capacity.

Theorem 1 confirms the reality of many service systems, in which providers allocate capacity to mitigate or eliminate negative interactions between different customer classes. An interesting observation in Theorem 1 is that any asymmetry in the classes’ willingness to pay for service (i.e., the value of \( a \)) does not affect the sufficient conditions. Although one might expect larger asymmetry to favor capacity allocation, one must also realize that a class willing to pay more for service than the other class is neither a reason to separate them nor a reason to keep them together. The aim of capacity allocation is only to prevent customer interactions that hurt the overall customer experience, which, of course, has nothing to do with their willingness to pay for the service. The provider eventually takes into account any asymmetry in the willingness to pay for service by letting more class-2 customers in the optimal customer mix through pricing or by allocating more capacity to them (when classes use different capacity segments).

Theorem 1 also has an important practical implication. The service provider only needs to know or guess the net appreciation between the classes to decide on her strategy. When the net appreciation is not negative, there is no reason to make any capacity allocation arrangements because blending the customers can only be beneficial. This is a consequence of the price discrimination that provides a great deal of simplicity in the decision making. The flexibility of charging different prices helps the service provider overcome any inherent disadvantage that allocation or non-allocation brings. In other words, the service provider can deal with the asymmetry in the willingness-to-pay through pricing, whether or not the capacity is allocated for the exclusive use of each class. As a result, parameter \( a \) plays no role in the service provider’s decision to go or not to go with capacity allocation.

\(^2\)In the statements of Theorems 1-2, note that the optimality of not allocating capacity does not necessarily imply an inclusive system; it implies that the provider cannot achieve strictly higher revenue by allocating capacity.
allocation. This is not the case when both classes have to be charged the same price. The single-price constraint makes the asymmetry parameter $a$ a significant factor, as the following theorem indicates.

**Theorem 2.** Suppose that the service provider cannot discriminate but has the flexibility to allocate capacity for the exclusive use of each class. Then, there exists $b^*(a)$ such that

(i) If $b_1 \leq 0$, $b_2 \leq 0$, it is optimal to allocate capacity.
(ii) If $b_2 > 0 > b_1$, then
   (a) If $b_1 \leq a - K/\Lambda$, it is optimal to allocate capacity.
   (b) If $b_1 > a - K/\Lambda$ and $b \leq b^*(a)$ (with $b^*(a) \geq 0$), it is optimal to allocate capacity.
   (c) If $b_1 > a - K/\Lambda$ and $b \geq b^*(a)$ (with $b^*(a) \geq 0$), it is optimal to not allocate capacity.
(iii) If $b_1 > 0 > b_2$, then
    (a) If $b_2 \leq -K/\Lambda$, or $b_1 \leq a - K/\Lambda$ and $b_2 > -K/\Lambda$, it is optimal to allocate capacity.
    (b) If $b_1 > a - K/\Lambda$, $b_2 > -K/\Lambda$ and $\lambda^*_1 = 0$ in problem (P2), it is optimal to allocate capacity.
    (c) If $b = 0$, $a/2 \geq b_1 > a - K/\Lambda$, $b_2 \geq -a/2$ and $\lambda^*_1 > 0$ in problem (P2), it is optimal to not allocate capacity.
(iv) If $b_1 \geq 0$, $b_2 \geq 0$, it is optimal to not allocate capacity.

Furthermore, if $b = b^*(a)$, allocating and not allocating capacity yield the same revenue to the provider.

As Theorem 2 shows, the provider’s decision regarding capacity allocation is more complicated if she cannot price discriminate. There are two important observations we can make by comparing Theorems 1 and 2. First, a single-price policy leads to capacity allocation in more cases than price discrimination does. Second, when deciding whether or not to allocate capacity, the ability to price discriminate allows the service provider to determine the optimal choice with less information on customer-mix effects. Specifically, whereas a price-discriminating service provider needs to know only the sign of the net appreciation term $b$, she needs to know individual terms $b_1$ and $b_2$ if she is going to charge both classes the same price. Nonetheless, notice that parts (i) and (iv) of Theorem 2 are analogous to parts (i) and (ii) of Theorem 1, respectively. If classes dislike each other, it is better to keep them separate. If there is mutual appeal, it is more profitable to refrain from capacity allocation. Thus, if class feelings are mutual, neither the pricing policy nor the asymmetry in the willingness to pay for service have an impact on whether or not capacity should be allocated to each class.

The provider’s choice is less straightforward if class perceptions go in opposite directions. When $a = 0$, the two cases are completely symmetric and parts (ii) and (iii) of Theorem 2 are essentially identical statements. If $b_2 > 0 > b_1$, the sufficient conditions of Theorem 2-(ii) confirm that the
single-price constraint results in capacity allocation in more cases than price discrimination does. If \( b_1 > 0 > b_2 \), and classes’ feelings toward each other are so different that the provider’s best choice without capacity allocation is an exclusive system as outlined in Lemma 2, then she is better off allocating capacity when there is such a flexibility (parts (iii)-a and (iii)-b of Theorem 2). What is more interesting is the case when conditions are such that the provider’s optimal choice is to run an inclusive system, i.e., not allocating capacity, and price discrimination is not an option (Theorem 2(ii)-c,(iii)-c and (iv)) and we investigate that in more detail in the next section using some numerical examples.

6. Numerical examples and Sensitivity analysis

This section is divided into four subsections. In the first subsection, we expand on our discussion of Theorems 1 and 2 using the results of a numerical study. Next, we highlight the importance of the customer mix effects in service systems. In the third subsection, we focus on the sensitivity of the different policies to the various parameters to check their robustness and to better understand their performance. Finally, we study whether our results continue to hold when we relax the assumption of the same class size for the two customer classes.

Comparison of the different policies for asymmetric classes

We use three different sets of parameters with zero net appreciation \((b = 0)\) to gain insights on the different revenues one might obtain depending on the capacity allocation decision and the pricing policy followed. We set \( \Lambda = 100 \), \( K = 80 \) and \( a = 0.5 \) and we summarize the optimal solutions in Table 1. As discussed earlier, when the manager can price discriminate, she will attract the same mix of customers, independently of the capacity decision, by charging different prices and yield the same revenue \((\lambda_1^* = 27.5, \lambda_2^* = 52.5 \text{ and } R(27.5, 52.5) = 71.125)\). However, in the single price policy,
mixing the customers or allocating capacity yields the same revenue only when $b_1 = b_2 = 0$ (S1). Note that the revenues are the same, but prices can be different for the two capacity allocation decisions. Suppose now that $b_1 = 0.3$ and $b_2 = -0.3$ so that $b_1 > 0 > b_2$. These changes do not affect the revenue under capacity allocation (because different classes do not interact), but they change the revenue of an inclusive system. Specifically, the new solution yields revenue $R(30,50) = 71$, which is higher than before, and thus operating an inclusive system with both classes sharing the whole capacity is strictly better than allocating capacity for the exclusive use of each class. There are two interesting points to highlight here. First, although asymmetry in customer-mix effects hurts revenue when classes are ex ante symmetric ($a = 0$), as we observe from this example, that may not be the case when classes are ex ante asymmetric ($a > 0$). Second, when customers are no longer indifferent about the presence of customers from the other class, it is strictly preferable to have a system where both classes share the whole capacity. As we explain in the following, both of these insights are consequences of the restriction to charge the same price to both classes.

If classes are ex ante asymmetric and $b_1 = b_2 = 0$, class 2 would pay more for service than class 1 if the provider could price discriminate. However, the single-price constraint requires that class–1 customers pay more (and that class–2 customers pay less) than what they would have had under price discrimination, thereby resulting in inefficient pricing. Suppose now that $b = 0$ but $b_1 > 0 > b_2$. In that case (and all else being equal), class–1 customers are willing to pay more than class–2 customers in order to be around customers of the other type; in other words, the effect of (small) asymmetry in class feelings is in line with the single-price mandate. What does this mean for the revenue of an inclusive system? Compared with the case where each class is indifferent about the other’s presence ($b_1 = b_2 = 0$), it is better to have a small asymmetry in perceptions, with class–1 customers having slight preference for having class–2 customers around while class–2 customers having slight preference for not having class–1 customers around. As a result, in S2, price discrimination has little benefit. Note, however, that if this asymmetry in perceptions is significant (S3 in Table 1), then we know that it will be the primary difficulty in implementing a single-price policy and will force the provider to separate the classes or allocate the whole capacity to one class. Also, if the asymmetry is in the other direction, with class–2 customers enjoying the presence of class–1 customers, class feelings are no longer in line with the single-price mandate and as a result, it is less likely for an inclusive system to be the service provider’s optimal choice.

Part (iii)-c of Theorem 2 states conditions that guarantee the optimality of capacity sharing, and these conditions are particularly interesting. As long as the net appreciation term remains at zero, a small asymmetry in classes’ feelings about each other increases the revenue of a system when classes are not separated, and this is stated in the Theorem but can also be observed in Table 1 (S2). Because it is strictly better to not separate classes if $b_1$ and $b_2$ are sufficiently small
in absolute value and $b = 0$, the provider would also be better off doing so for small yet negative values of $b$. This means that in some cases, a single-price policy makes capacity allocation less likely than price discrimination does. This might appear to be contradicting one of the insights we have obtained so far, i.e., that single-price policies lead to more exclusivity. It is true that if the provider’s choice is only between an inclusive system where customers share the whole capacity and an exclusive system where customers from only one class are admitted, then a single-price mandate always leads to more exclusivity because that mandate disappears in an exclusive system. However, exclusivity as a result of separating the two customer classes by capacity allocation does not make the single-price mandate disappear. In that case, there might be some benefit from keeping ex ante asymmetric classes together and mitigating the pricing inefficiency that a single price causes, even if these classes feel differently about being around each other and their net appreciation is negative.

**Value of capturing customer mix effects**

To further highlight the value of capturing the customer mix effects, we compare the optimal revenues with the revenues we would have achieved had we ignored the parameters $b_1$ and $b_2$ by assuming $b_1 = b_2 = 0$. We will use the examples in the previous section to make this comparison. S1 in Table 1 provides the prices to make such a comparison. In S2 when customers can share the service facility and there is price discrimination (CS-DP), the revenue would be $R'(41.5, 36.5) = 67.75$ instead of $R(27.5, 52.5) = 71.125$. When both classes pay the same price (CS-SP), that is $p = 0.85$, then the revenue would be $R'(32.7, 47.3) = 68$ instead of the optimal $R(30, 50) = 71$. Similarly, for the set of parameters in S3, under price discrimination (CS-DP), the revenue would drop to $R'(0, 52.5) = 51.19$ compared to $R(27.5, 52.5) = 71.125$. In this case, ignoring the customer mix effects forces the system to become an exclusive one due to the high price charged to class-1 customers. For the single price policy (CS-SP), the revenue would be $R'(40.4, 31.4) = 60.95$ instead of $R(45, 35) = 65$. These examples are indicative of how high the losses can be and also confirm the fact that a suboptimal capacity allocation strategy might be followed. Taking into consideration that these losses become higher and more discernible when $b \neq 0$ further supports the importance of a correct capacity allocation decision and pricing strategy.

**Sensitivity analysis with respect to system capacity and the strength of customer asymmetry and interaction effects**

We have also conducted numerical studies to understand the impact of the parameters $a$, $b$, $K$ on the revenue for the different policies and we show some interesting cases in Figure 3. Starting with a negative net appreciation, capacity allocation is superior to mixing the customers and as $a$ increases, chances are higher to operate an exclusive system at least for low capacity; a small
facility can be filled up with class-2 customers who pay more. As $b$ increases, mixing the customers becomes more profitable with $b = 0$ making the policies equivalent and $b > 0$ making capacity sharing the preferred policy. When the manager decides to mix the customers, she has definitely the incentive to convince the two classes to enjoy each other’s company to achieve a higher revenue. Naturally, price discrimination is at least as good as single price policy which can possibly give the incentive to the provider to price discriminate, even if it is not legal, and incur a penalty. Using an example from the figure with $a = 1$ (when $a = 0$, the pricing strategy does not matter) and $b = -0.5$, $K = 150$, the manager can achieve 25% higher revenues if she charges differently the
Table 2  Optimal Revenue under the four different policies as the class sizes changes when \( K = 80 \), \( b = 0 \), \( a = 1 \) and \( \Lambda_{3-i} = 100 \).

<table>
<thead>
<tr>
<th>( \Lambda_i )</th>
<th>CA-SP</th>
<th>CA-DP</th>
<th>CS-DP</th>
<th>CS-SP</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 1 )</td>
<td>50</td>
<td>96</td>
<td>99</td>
<td>72.3</td>
</tr>
<tr>
<td>( i = 2 )</td>
<td>100</td>
<td>96</td>
<td>100.5</td>
<td>100.5</td>
</tr>
<tr>
<td>( i = 3 )</td>
<td>150</td>
<td>96</td>
<td>117.3</td>
<td>101.4</td>
</tr>
<tr>
<td>( i = 4 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

two classes (Figure 3(c)). A last but important observation from Figure 3 is that increasing the capacity can benefit the facility up to a certain value but afterwards the revenue is almost constant and therefore, investing in larger capacity might not be worth it.

Different class sizes

Heretofore, we focused on leisure facilities that attract two customer classes of the same size but this might not be always the case. One interesting fact we observed from our numerical experiments is that for high values of \( a \) the changes in the size of class 2 have a higher impact on the revenue than changes in \( \Lambda_1 \). This is due to the asymmetry of the two classes in terms of their willingness to pay for the service (Table 2). Moreover, we observed that under price discrimination, as the size of one class increases, while the other is constant or decreasing, the system tends to operate in an exclusive manner more often than before (\( b^*(K) \) is higher). But if \( a \) is higher and class 2 is small, i.e. the high value customers are few, then it is not worth it to operate exclusively and the facility is better off admitting a mix of customers. In other words, the manager has to exhaust her options of attracting the high value customers but might be limited by their class size. It is also important to note that our numerical analysis suggests that the results of Theorem 1 continue to hold. Not surprisingly, however, the conditions of Theorem 2 have to be modified to account for different class sizes.

7. Conclusions

This paper deals with a particular type of service setting, where service takes an extended period of time and is shared by others so that what happens during service or more specifically who else is there during service is a very important determinant of the customers’ utility. Despite the prevalence of such services in practice, these features are sometimes ignored by the service managers and they have received limited attention in the operations literature. One of the important contributions of this paper is the development of a stylized framework that can be helpful in building new models to investigate various research questions (e.g., effects of competition) regarding shared service systems.

We used the framework we developed to provide insights into the use of pricing and capacity allocation as levers to control the customer mix and crowding. Some of our findings conform
to what we observe in practice and our intuition as to what a profit–maximizing service provider should be doing (for example, the use of discounts if there is asymmetry between how different classes feel about each other), whereas others are either counter–intuitive or help us gain a deeper understanding of some of the issues for which intuition does not provide much help. For example, we find that if the service provider is restricted to charge the same price to both customer classes and if the two classes are highly asymmetric (either with respect to mutual appreciation or willingness-to-pay), the only way for the service provider to make money could be by offering the service to only one class. Interestingly, however, when there is mutual dislike between the two classes, the facility can profitably operate an inclusive service system serving both classes. In short, when faced with sufficiently asymmetric customer classes, the only way for the service provider to survive is by restricting access to a particular class of customers or by allocating different portions of its capacity for the exclusive use of different customer classes. Thus, strong asymmetry requires some sort of discrimination or capacity allocation for the survival of the firm.

For a service provider who can use price discrimination, the choice between allocating capacity for the exclusive use of different classes and making the whole capacity available to all its customers depends purely on the net appreciation between classes, not on crowding effects, capacity, or the degree of asymmetry in the two customer classes’ willingness to pay—if there is any. Specifically, capacity allocation is desirable if the net appreciation is negative. If price discrimination is not an option, capacity allocation could be desirable even if the net appreciation is positive. Thus, in many cases, disallowing price discrimination makes it more likely for firms to choose capacity allocation to different customer classes for their exclusive use. It is, however, possible that restriction to a single-price policy might lead the provider to switch to an inclusive system with the whole capacity available to both classes. This can only happen if the class with the lower willingness to pay for service likes the other class because only in this case, inclusivity helps reduce the gap between the willingness-to-pay of the two classes.

Our results highlight the importance of having a deeper understanding of customer-mix effects on the utilities of different customer classes, because they are highly relevant in choosing the pricing and capacity allocation policies to be employed. Many articles in the marketing literature have established the presence and importance of these effects, but we are not aware of any work that has aimed to quantify them. To take advantage of the insights, a rough estimate of the parameters might sometimes be sufficient to determine the right strategy. However, some quantification of the customer mix effects, i.e. the sign of \( b \) and/or which effect is dominant could be critical in maximizing profit. Thus, one avenue for future research is to develop a framework that can be utilized in measuring customer-mix effects empirically in different service settings. Capturing the valuation for the service is also challenging, yet necessary, to determine the optimal pricing policy. In
this direction, economists and marketing researchers have used surveys, experiments, transactions
data to find good approximations for the willingness to pay of the customers (Wertenbroch and
Skiera 2002). Most of these methods can be put into use when estimating customer mix effects.

In some of the service settings we have discussed, the service establishment can in fact gain some
pooling benefit if it allows the two customer classes to share the facility (or possibly incur a cost to
separate the physical space). This is something we ignore in our formulation. If this benefit were
to be considered, our results would change accordingly; the threshold on the customer mix effects
would be negative for the capacity allocation to be optimal accounting for the pooling loss. As
expected, the new threshold would depend on the actual cost savings from pooling resources; the
higher the saving, the lower the threshold would be. In other words, when the savings from pooling
is higher, the classes’ appreciation of each other would need to be stronger in the negative direction
for capacity allocation to be optimal. In some cases, changing the capacity allocation strategy might
be costly as it may require rebuilding the whole facility. In that case, the problem is more complex
and its analysis would require a formulation that is different from the one we considered in this
paper. If rebuilding the facility is an option to the provider, i.e. she is not restricted by the actual
size of the facility, then at the beginning of the time horizon, she has to take into consideration
several factors including the size of the investment, the competition, the market targeted etc. and
investigate how much profit the firm would make at different levels of capacity investment in order
to make an optimal decision.

Our model assumed that there is no demand uncertainty and customers make a decision to join or
not join simultaneously knowing how all the other customers will behave. It would however also be
interesting to consider a formulation where demand is uncertain, customers arrive sequentially and
make decisions as they arrive, and the manager has the option to dynamically change the admission
price. Another interesting extension would be to consider multiple facilities in competition each
offering different capacity arrangements to their customers.

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Technical Appendix

Proof of Lemma 1 We first show part (i). Note that solution \((0, \epsilon)\) is never optimal. Consider solution \((0, \epsilon)\) with \(\epsilon > 0\) and sufficiently small. This solution yields revenue \(R(0, \epsilon) = \epsilon(1 + c(\epsilon/K)) - \epsilon^2/\Delta\). Now \(\lim_{\epsilon \to 0} R(0, \epsilon) > \lim_{\epsilon \to 0} \epsilon - \epsilon^2/\Delta = 0\) (recall that \(c(0) > -1\)). Therefore, \(\lambda_1^* = \lambda_2^* < 0\).

To show that \(\lambda_1^* \lambda_2^* > 0 \Rightarrow \lambda_1^* = \lambda_2^*\), consider problem \((P1)\) and note that \(R(\lambda_1, \lambda_2) = \lambda_1 \{1 - \lambda_1 - b_1 \lambda_2 / (\lambda_1 + \lambda_2) + c[(\lambda_1 + \lambda_2) / K]\} + \lambda_2 \{1 + a - \lambda_2 / (\lambda_1 + \lambda_2) + \lambda_1 (\lambda_1 + \lambda_2) / K\}\}. Let \(\{\mu_i \geq 0 : i = 1, 2, 3\}\) be KKT multipliers for constraints \(-\lambda_1 + \lambda_2 + K \geq 0, \lambda_1 \geq 0, \lambda_2 \geq 0\), respectively. A candidate optimal solution must satisfy \(\mu_1 (-\lambda_1 - \lambda_2 + K) = 0, \mu_2 \lambda_1 = 0, \mu_3 \lambda_2 = 0\), and the following stationarity conditions,

\[
1 - 2\lambda_1 / (\lambda_1 + \lambda_2) - \lambda_1 + \mu_2 + c[(\lambda_1 + \lambda_2) / K] / K = 0, \tag{8}
\]

\[
1 + a - 2\lambda_2 / (\lambda_1 + \lambda_2) - \lambda_2 + \mu_3 + c[(\lambda_1 + \lambda_2) / K] / K = 0. \tag{9}
\]

Let \(a = 0, \mu_1 = \mu_2 = \mu_3 = 0\), and subtract equation (8) from equation (9), which yields

\[
(\lambda_1 - \lambda_2) / (2/\Delta + b/\lambda_1 + \lambda_2) = 0.
\]

Therefore, either \(\lambda_1 = \lambda_2\) or \(\lambda_1 + \lambda_2 = -b\Delta/2\), which implies \(b < 0\). Likewise, letting \(a = 0, \mu_1 = \mu_2 = \mu_3 = 0\), and adding up equations (8) and (9) yields \(\lambda_1 \lambda_2 = (1 + b + c(-b\Delta/(2K)) - b\lambda_1 / (2K)c(-b\lambda_1/(2K))\lambda_2^2/4\), which in turn implies \(1 + b + c(-b\lambda_1/(2K)) - b\lambda_1 / (2K)c(-b\lambda_1/(2K)) < 0\). Substituting \(\lambda_1 + \lambda_2 = -b\Delta/2\) into (8) leads to \(1 + b + c(-b\lambda_1/(2K)) - b\lambda_1 / (2K)c(-b\lambda_1/(2K)) = -2\lambda_1 (1 + 2\lambda_2 / (b\lambda)) / \Delta\), which is negative. This implies that \(1 + 2\lambda_2 / (b\lambda) > 0\). Similarly, if we substitute \(\lambda_1 + \lambda_2 = -b\Delta/2\) into (9), we have that \(1 + 2\lambda_1 / (b\lambda) > 0\). Adding these two implies that \(1 + 2(\lambda_1 + \lambda_2) / (b\lambda) > 0\) that leads to a contradiction since we assumed that \(\lambda_1 + \lambda_2 = -b\Delta/2\). Therefore, \(\lambda_1^* \lambda_2^* > 0 \Rightarrow \lambda_1^* = \lambda_2^*\).

Part (ii) follows directly from the fact that \(R(\lambda, 0) = R(0, \lambda)\) if \(a = 0\).

We now show part (iii). First, note that \(\lambda_1 = \lambda_2\) cannot be optimal if \(a > 0\), because the system of equations (8)–(9) does not have a solution in that case. Now consider solution \((x, y, \Delta x)\), \(x > y\). This solution yields revenue \(R(x, y) = x(1 - x / \Delta x) + y(1 + a - y / \Delta x) + bx y / (x + y) + (x + y) e((x + y) / K)\). Likewise, \(R(y, x) = y(1 - y / \Delta x) + x(1 + a - x / \Delta x) + bx y / (x + y) + (x + y) e((x + y) / K)\). Note that \(R(y, x) = R(x, y) = a(x - y) > 0\).

Thus, we must have \(\lambda_2^* > \lambda_1^*\) if \(a > 0\). \(\square\)

Proof of Proposition 1 We use the same notation as in the proof of Lemma 1. First, recall that solution \((0,0)\) is never optimal. Second, we know from Lemma 1 that \(\lambda_2^* \geq \lambda_1^* \geq 0\), therefore, \(\lambda_2^* > 0\), and thus, \(\mu_3^* = 0\) always. Hereafter, we remove the complementary slackness constraint \(\mu_3 \lambda_2 = 0\) from further consideration.

To show part (i), consider a feasible solution to \((P1)\) where \(\lambda_1 > 0\). Note that for sufficiently negative values of \(b\) such a solution cannot be optimal because \(\lim_{b \to -\infty} R(\lambda_1, \lambda_2) = -\infty\) if \(\lambda_1 > 0\). Therefore, for sufficiently negative values of \(b\), \(\lambda_1^* = 0\). Similarly, for sufficiently positive values of \(b\) we must have \(\lambda_1^* > 0\) because \(\lim_{b \to \infty} R(\lambda_1, \lambda_2) = \infty\) if \(\lambda_1 > 0\). Suppose now that \(\lambda_1^* > 0, \lambda_2^* = \lambda^* \text{ at } b = b^*\), but \(\lambda_1^* = 0, \lambda_2^* = \lambda^* \text{ at } b = b^* > b^*\). However, by the Envelope Theorem, \(\partial R(\lambda_1^*, \lambda^*) / \partial b = \lambda_1^* \lambda^*/(\lambda_1^* + \lambda^*) > 0\), whereas \(\partial R(0, \lambda^*) / \partial b = 0\). Therefore,
solution \((\lambda_1^*, \lambda^*)\) could not have been optimal at \(b = b^*\), and we have arrived at a contradiction. We thus conclude that there exists threshold \(b^*\), which depends on \(K\) in general, such that \(\lambda_1^* = 0\) if \(b \in (-\infty, b^*(K)]\), and \(\lambda_1^* > 0\) if \(b \in (b^*(K), \infty)\).

For the proof of part (ii), it suffices to show that function \(R(\lambda_1^*, \lambda_2^*)\uparrow K\) if \(\lambda_1^* + \lambda_2^* = K\) and \(K \leq \min(\Lambda[1 + a + c(1)]/2[1 + c(1)]\Lambda/3)\). By the Envelope Theorem, \(\partial R(\lambda_1^*; \lambda_2^*)/\partial K = \mu_1^* - (\lambda_1^* + \lambda_2^*)^2c'[(\lambda_1^* + \lambda_2^*)/K]/K^2\), where \(\mu_1^*\) can be calculated using equation (9). If \(\lambda_1^* = 0\), then \(\mu_1^* = 1 + a + c(1) + c'(1) - 2K/\Lambda\); thus, \(\partial R(\lambda_1^*; \lambda_2^*)/\partial K = 1 + a + c(1) - 2K/\Lambda\); therefore, \(R(\lambda_1^*; \lambda_2^*) \uparrow K\) if \(K \leq \Lambda[1 + a + c(1)]/2\). If \(\lambda_1^* > 0\), then \(\partial R(\lambda_1^*; \lambda_2^*)/\partial K = 1 + a + c(1) - 2\lambda_2^*/\Lambda + b(K - \lambda_2^*)^2/K^2\). In addition, equations (8) and (9) jointly yield the solution \(\lambda_1^* = K/2 - a\Lambda/[2(2K + b\Lambda)]\), \(\lambda_2^* = K/2 + aK\Lambda/[2(2K + b\Lambda)]\); therefore, we require that \(b > a - 2K/\Lambda \geq -2K/\Lambda\) so that \(\lambda_1^* > 0\). (If \(b < a - 2K/\Lambda\), the solution in question is a local minimum.)

Because \(\lambda_2^* \geq K/2\) and \(\partial R(\lambda_1^*; \lambda_2^*)/\partial K\) is a second-order polynomial wrt \(\lambda_2^*\), for our purposes it suffices to show that \(\lim_{\lambda_2^* \to K^-} -\partial R(\lambda_1^*; \lambda_2^*)/\partial K \geq 0\) and that \(\partial R(\lambda_1^*; \lambda_2^*)/\partial K|_{\lambda_2^* = K/2} \geq 0\). To that end, first note that \(\lim_{\lambda_2^* \to K^-} -\partial R(\lambda_1^*; \lambda_2^*)/\partial K = 1 + a + c(1) - 2K/\Lambda \geq 0\), where the last inequality is because \(K \leq \Lambda[1 + a + c(1)]/2\).

Furthermore, \(\partial R(\lambda_1^*; \lambda_2^*)/\partial K|_{\lambda_2^* = K/2} = b/4 - K/\Lambda + 1 + c(1) > -3K/(2\Lambda) + 1 + c(1) \geq 0\), where the last two inequalities are because \(b > a - 2K/\Lambda\) and \(K \leq 2[(1 + c(1)]/3\), respectively.

Part (iii) follows immediately from the fact that \(\lambda_1^* + \lambda_2^* \leq 2\Lambda\).

To show part (iv), first note that parts (ii) and (iii) establish that there exists at least one switching point at which the system goes from being full to being not full. We show next that if the conditions of part (iv) hold, the switching point is unique. Letting \(\lambda_1 + \lambda_2 = K\) in equation (9) yields \(\mu_1^* = 1 + a + c(1) + c'(1) - 2\lambda_2^*/\Lambda + b(\lambda_2^*)^2/K^2\). To show that the switching point is unique, it suffices to show that \(\mu_1^*(K) / K\) is that to end, first suppose that \(b < 0\) \(\Rightarrow\) \(\lambda_1^* = 0\) \(\forall K > 0\). In this case, \(\mu_1^*(K) = 1 + a + c(1) + c'(1) - 2K/\Lambda \Rightarrow\) \(\mu_1^*(K) / K\) is that to end, first suppose that \(b < 0\) \(\Rightarrow\) \(\lambda_1^* > 0\) \(\forall K > 0\). Next, suppose \(b > 0\) \(\Rightarrow\) \(\lambda_1^* > 0\) \(\forall K > 0\). Letting \(\lambda_1 + \lambda_2 = K\), \(\mu_2 = 0\) in the system of equations (8)–(9) yields the solution \(\lambda_1^* = K/2 - a\Lambda/[2(2K + b\Lambda)]\), \(\lambda_2^* = K/2 + a\Lambda/[2(2K + b\Lambda)]\). Straightforward calculus yields \(\partial \mu_1^*(K)/\partial K = 1 - \Lambda - ba2^2\Lambda^3/(2K + b\Lambda)^3 < 0\).

**Proof of Lemma 3** The proof of part (ii) is straightforward. We show here the proof for part (i); in particular, we will show that \(\lambda_1^* \geq \lambda_2^*\), if \(b \geq b_{\text{in}}\), and \(\lambda_1^* \geq \lambda_2^*\) > 0. To that end, consider problem (P2') and note that \(R(\lambda_1, \lambda_2) = \lambda_1(1 - \lambda_1/\Lambda + b\lambda_2/(\lambda_1 + \lambda_2) + c[(\lambda_1 + \lambda_2)/K] + \lambda_2(1 - \lambda_2/\Lambda + b\lambda_1/(\lambda_1 + \lambda_2) + c[(\lambda_1 + \lambda_2)/K] + a\))\) \(\{\mu_1 \geq 0: i = 1, 2, 3\}\) be KKT multipliers for constraints \(-\lambda_1 + \lambda_2 + K \geq 0\), \(\lambda_1 \geq 0\), \(\lambda_2 \geq 0\), respectively. Let \(g \in \mathbb{R}\) be the Lagrange multiplier for constraint \([b_1/(\lambda_1 + \lambda_2) + 1/\Lambda] - b_2/(\lambda_1 + \lambda_2) + 1/\Lambda] \lambda_1 - a = 0\). A candidate optimal solution must satisfy \(\mu_1(\lambda_1 - \lambda_2 + K) = 0\), \(\mu_2(\lambda_1 - \lambda_2 + K) = 0\), \(\mu_3 \lambda_2 = 0\), \([b_1/(\lambda_1 + \lambda_2) + 1/\Lambda] - b_2/(\lambda_1 + \lambda_2) + 1/\Lambda] \lambda_1 - a = 0\), and the following stationarity conditions.

\[
1 - 2\lambda_1/\Lambda + b\lambda_2^2/(\lambda_1 + \lambda_2)^2 - \mu_1 + \mu_2 + c[(\lambda_1 + \lambda_2)/K] + (\lambda_1 + \lambda_2)c'[(\lambda_1 + \lambda_2)/K]/K

\[
- g(b\lambda_2/(\lambda_1 + \lambda_2)^2 + 1/\Lambda) = 0, \tag{10}
\]

\[
1 + a - 2\lambda_2/\Lambda + b\lambda_1^2/(\lambda_1 + \lambda_2)^2 - \mu_1 + \mu_3 + c[(\lambda_1 + \lambda_2)/K] + (\lambda_1 + \lambda_2)c'[(\lambda_1 + \lambda_2)/K]/K

\[
+ g(b\lambda_1/(\lambda_1 + \lambda_2)^2 + 1/\Lambda) = 0. \tag{11}
\]

Let \(a = 0\), \(\mu_1 = \mu_2 = \mu_3 = 0\), and subtract equation (10) from equation (11), which yields

\[
(\lambda_1 - \lambda_2 + g)[2/\Lambda + b/(\lambda_1 + \lambda_2)] = 0.
\]
Therefore, either $g = \lambda_2 - \lambda_1$ or $\lambda_1 + \lambda_2 = -b A / 2$. If $b_1 = b_2$, we know from part (i) of Lemma 1 that $\lambda_1^* = \lambda_2^*$, which remains an optimal solution here because it satisfies the single-price constraint (4) if $a = 0$. Thus, we assume—without loss of generality—that $b_1 > b_2$ for the remainder of the proof. Suppose that $\lambda_1 + \lambda_2 = -b A / 2$. Then, the last equation together with equation (4) for $a = 0$ yield $\lambda_1 + \lambda_2 = 0$, which is not possible unless $\lambda_1 = \lambda_2 = 0$. However, we know from part (i) of Lemma 1 that solution (0, 0) is never optimal. Therefore, $g^* = \lambda_2^* - \lambda_1^*$ when $\lambda_1^* \lambda_2^* > 0$ and $a = 0$.

Ignoring the non-binding constraints, the Lagrange function for problem (P2′) and $a = 0$ is $L(\lambda_1, \lambda_2) = \lambda_1 [1 - \lambda_1 / A + b_1 \lambda_2 / (\lambda_1 + \lambda_2) + c((\lambda_1 + \lambda_2) / K)] + \lambda_2 [1 - \lambda_2 / A + b_2 \lambda_1 / (\lambda_1 + \lambda_2) + c((\lambda_1 + \lambda_2) / K)] + g[\lambda_1 / (\lambda_1 + \lambda_2) + 1 / A] \lambda_2 - [b_2 / (\lambda_1 + \lambda_2) + 1 / A] \lambda_1 - a$. Let $L^*(\lambda_1^*, \lambda_2^*) \equiv \max L(\lambda_1, \lambda_2)$. By the Envelope Theorem, $\partial L^*/\partial b_2 = \lambda_1^* \lambda_2^*/(\lambda_1^* + \lambda_2^*) - g^* \lambda_1^*/(\lambda_1^* + \lambda_2^*) = (\lambda_1^*)^2/(\lambda_1^* + \lambda_2^*)$, where the second equality is because $g^* = \lambda_2^* - \lambda_1^*$. Likewise, $\partial L^*/\partial b_1 = (\lambda_2^*)^2/(\lambda_1^* + \lambda_2^*)$.

To complete the proof, suppose wlog that $b_1 > b_2$ and that solution $(\lambda_1^*(b_1), \lambda_2^*(b_1))$, where $\lambda_1^* < \lambda_2^*$, satisfies constraint (4) and the stationarity conditions (10)-(11). By applying the expressions that the Envelope Theorem stipulates, $R(\lambda_1^*(b_1), \lambda_2^*(b_1)) \uparrow b_1$; therefore, solution $(\lambda_1^*(b_1), \lambda_2^*(b_1))$, where $0 < \lambda_1^*(b_1) < \lambda_2^*(b_1)$, will be optimal if $b_1$ is sufficiently large. Note, however, that if $b_1 > b_2 \geq 0$ or $b_1 \geq 0 > b_2$ and $\lambda_2 > \lambda_1$, constraint (4) cannot be satisfied. Therefore, we have reached a contradiction; a solution such that $\lambda_1 < \lambda_2$ cannot be optimal if $b_1 \geq b_2$. □

**Proof of Proposition 2** We use the same notation as in the proof of Lemma 3. First, note that solution (0, 0) is never optimal. Second, we know from part (iii) of Lemma 1 that in an exclusive system, $\lambda_2^* > \lambda_1^* = 0$ if $a > 0$. If $a = 0$, $\lambda_2^* > \lambda_1^* = 0$ is still optimal. Therefore, $\lambda_2^* > 0$ and thus, $\mu^*_2 = 0$ always. Hereafter, we remove the complementary slackness constraint $\mu_2 \lambda_2 = 0$ from further consideration.

To prove the first statement of part (i), consider a feasible solution to (P2′) such that $\lambda_1 > 0$. Note that for sufficiently negative values of $b$ such a solution cannot be optimal because $\lim_{b \to -\infty} R(\lambda_1, \lambda_2) = -\infty$ if $\lambda_1 > 0$. Therefore, for sufficiently negative values of $b$, $\lambda_1^* = 0$. Similarly, for sufficiently positive values of $b$ we must have $\lambda_1^* > 0$ because $\lim_{b \to +\infty} R(\lambda_1, \lambda_2) = +\infty$ if $\lambda_1 > 0$.

Ignoring the non-binding constraints, the Lagrange function for problem (P2′) is $L(\lambda_1, \lambda_2) = \lambda_1 [1 - \lambda_1 / A + b_1 \lambda_2 / (\lambda_1 + \lambda_2) + c((\lambda_1 + \lambda_2) / K)] + \lambda_2 [1 - \lambda_2 / A + b_2 \lambda_1 / (\lambda_1 + \lambda_2) + c((\lambda_1 + \lambda_2) / K)] + g[\lambda_1 / (\lambda_1 + \lambda_2) + 1 / A] \lambda_2 - [b_2 / (\lambda_1 + \lambda_2) + 1 / A] \lambda_1 - a$. Let $L^*(\lambda_1^*, \lambda_2^*) \equiv \max L(\lambda_1, \lambda_2)$. By the Envelope Theorem, $\partial L^*/\partial b_1 = (\lambda_2^*)^2/(\lambda_1^* + \lambda_2^*)$. First, we will show that $\lambda_1^* + g^* \geq 0$. Suppose, to the contrary, that $\lambda_1^* + g^* < 0$ at the optimal solution. This necessarily implies that for this particular solution $R(\lambda_1^*, \lambda_2^*) \downarrow b_1$; $b_1 \in (-\infty, \infty)$. Now recall that if $\lambda_1 > 0$, $\lim_{b \to +\infty} R(\lambda_1, \lambda_2) = -\infty$. Therefore, a solution such that $\lambda_1^* + g^* < 0$ cannot be optimal at any value of $b_1$. By the same logic, it must be true that $\lambda_2^* - g^* \geq 0$ at the optimal solution. Therefore, $R(\lambda_1^*, \lambda_2^*) \uparrow b_1$ and $R(\lambda_1^*, \lambda_2^*) \uparrow b_2$. On the other hand, the revenue of an exclusive system is not a function of $b_1$, $b_2$; therefore, we conclude that there exists threshold $b^*$, which depends on $K$ in general, such that $\lambda_1^* = 0$ if $b \in (-\infty, b^*(K))$, and $\lambda_1^* > 0$ if $b \in (b^*(K), \infty)$. This completes the proof of the first statement of part (i).

To prove the second statement of part (i), let $a = 0$, $b_1 + b_2 = \bar{b}$, i.e., $b$ is constant, and assume wlog that $b_1 \geq b_2$, in which case $\lambda_1^* \geq \lambda_2^*$ by Lemma 3. Also, recall from the proof of Lemma 3 that $g^* = \lambda_2^* - \lambda_1^*$ when
\( \lambda_1^* \lambda_2^* > 0 \) and \( a = 0 \). Then, \( \partial \mathcal{L}(\lambda_1^*, \lambda_2^*)/\partial b_1 - \partial \mathcal{L}(\lambda_1^*, \lambda_2^*)/\partial b_2 = [\lambda_2^* - (\lambda_1^* \lambda_2^*)^2]/(\lambda_1^* + \lambda_2^*) \leq 0 \). Therefore, \( \mathcal{L}(\lambda_1^*, \lambda_2^*) \downarrow \Delta b \). Because the revenue of an exclusive system is not a function of \( b_1, b_2 \), there must exist threshold \( \Delta b^* \), which depends on \( K \) in general, such that \( \lambda_1^* > 0 \) if \( b \in (0, \Delta b^*(K)) \), and \( \lambda_1^* = 0 \) if \( \Delta b \in (\Delta b^*(K), \infty) \).

For the proof of part (ii), it suffices to show that function \( R(\lambda_1^*, \lambda_2^*) \uparrow K \) if \( \lambda_1^* + \lambda_2^* = K \) and \( K \leq \min(\Lambda[1 + c(1)], \Lambda[1 + a + c(1)]/2) \). By the Envelope Theorem, \( \partial R(\lambda_1^*, \lambda_2^*)/\partial K = \mu_1^* - (\lambda_1^* + \lambda_2^*)^2 c'(\lambda_1^* + \lambda_2^*)/K^2 \), where \( \mu_1^* \) can be calculated using equation (11). If \( \lambda_1^* = 0 \), then \( \mu_1^* = 1 + a + c(1) + c'(1 - 2 K/\Lambda) \); thus, \( \partial R(\lambda_1^*, \lambda_2^*)/\partial K = 1 + a + c(1) - 2 K/\Lambda \); therefore, \( R(\lambda_1^*, \lambda_2^*) \uparrow K \) if \( K \leq \Lambda[1 + a + c(1)]/2 \). Consider next the case \( \lambda_1^* > 0 \). The fact that \( \lambda_1^* + \lambda_2^* = K \) and equation (4) jointly imply the solution \( \lambda_1^* = K[K + (b_1 - a)\Lambda]/(2 K + b\Lambda) \), \( \lambda_2^* = K[K + (b_2 + a)\Lambda]/(2 K + b\Lambda) \); therefore, we require that \( b_1 > a - K/\Lambda \), \( b_2 < -K/\Lambda \), \( b > a - 2 K/\Lambda \geq -2 K/\Lambda \) so that \( \lambda_1^* > 0 \). Using the expressions for \( \lambda_1^*, \lambda_2^* \), and subtracting equation (10) from (11) yields \( g^* = (a + b_2 - b_1) K/\Lambda/(2 K + b\Lambda) \). Also, by equation (11), \( \mu_1^* = 1 + a + c(1) + c'(1 - 2 K/\Lambda + b(K - \lambda_2^*)^2/K^2 + g^*[b(K - \lambda_2^*)/K^2 + 1/\Lambda] \); thus, \( \partial R(\lambda_1^*, \lambda_2^*)/\partial K = 1 + a + c(1) - 2 K/\Lambda + b(K - \lambda_2^*)^2/K^2 + g^*[b(K - \lambda_2^*)/K^2 + 1/\Lambda] \). Because \( K > \lambda_2^* > 0 \) and \( \partial R(\lambda_1^*, \lambda_2^*)/\partial K \) is a second-order polynomial wrt \( \lambda_2^* \), for our purposes it suffices to show that \( \lim_{\lambda_2^* \to K} \partial R(\lambda_1^*, \lambda_2^*)/\partial K \geq 0 \) and that \( \lim_{\lambda_2^* \to a} \partial R(\lambda_1^*, \lambda_2^*)/\partial K \geq 0 \). Because \( \lim_{\lambda_2^* \to K} g^* = (b_2 + K/\Lambda) K/(2 K + b\Lambda) \), \( \lim_{\lambda_2^* \to K} \partial R(\lambda_1^*, \lambda_2^*)/\partial K = 1 + a + c(1) - 2 K/\Lambda + (b_2 + K/\Lambda) K/(2 K + b\Lambda) \geq 0 \), where the last inequality is because \( K \leq \Lambda[1 + a + c(1)]/2 \) and \( b_2 > -K/\Lambda \). Similarly, \( \lim_{\lambda_2^* \to 0} g^* = -K \), because \( b_1 > -K/\Lambda \), \( \lambda_2^* \to 0^+ \) if \( b_2 < -K/\Lambda \) and \( a \to 0 \). Thus, \( \lim_{\lambda_2^* \to 0} \partial R(\lambda_1^*, \lambda_2^*)/\partial K = 1 + c(1) - K/\Lambda \geq 0 \), because \( K \leq \Lambda[1 + c(1)] \).

Part (iii) follows immediately from the fact that \( \lambda_1^* + \lambda_2^* \leq 2 \Lambda \).

To show part (iv), first note that parts (ii) and (iii) establish that there exists at least one switching point at which the system goes from being full to being not full. We show next that if the conditions of part (iv) hold, the switching point is unique. The proof for the case \( b << 0 \) can be found in the proof of part (iv) of Proposition 1. Next, suppose that \( b >> 0 \Rightarrow \lambda_1^* > 0 \forall K \geq 0 \). The fact that \( \lambda_1^* + \lambda_2^* = K \) and equation (4) jointly imply the solution \( \lambda_1^* = K[K + (b_1 - a)\Lambda]/(2 K + b\Lambda) \), \( \lambda_2^* = K[K + (b_2 + a)\Lambda]/(2 K + b\Lambda) \). Using these expressions for \( \lambda_1^*, \lambda_2^* \), and subtracting equation (10) from (11) yields \( g^* = (a + b_2 - b_1) K\Lambda/(2 K + b\Lambda) \). Also, by equation (10), \( \mu_1^* = 1 + c(1) + c'(1 - 2 K/\Lambda + b(K - \lambda_2^*)^2/K^2 + g^*[b(K - \lambda_2^*)/K^2 + 1/\Lambda] \). To show that the switching point is unique, it suffices to show that \( \mu_1^*(K) \downarrow K \). To that end, note that

\[
\frac{\partial \mu_1^*(K)}{\partial K} = \frac{-8 K^3 - 6 b K \Lambda (2 K + b\Lambda) - 2 b^2 \Lambda (a(b_1 - b_2) + 2 b_1 b_2)}{\Lambda (2 K + b\Lambda)^3}.
\]

In the last fraction, the denominator is positive because \( b > 0 \); thus, we focus on the numerator, whose sign is ambiguous in general. First, notice that the numerator is strictly decreasing in \( K \), because \( b > 0 \). Second, notice that if \( a(b_1 - b_2) + 2 b_1 b_2 \geq 0 \), then \( \partial \mu_1^*(K)/\partial K < 0 \), which implies that the switching point is unique. Suppose next that \( a(b_1 - b_2) + 2 b_1 b_2 < 0 \). Then, for sufficiently small values of \( K \), \( \partial \mu_1^*(K)/\partial K > 0 \); therefore, as the numerator is strictly decreasing in \( K \), there exists a unique value of \( K \) above which \( \partial \mu_1^*(K)/\partial K < 0 \). In addition, because \( \lambda_1^* + \lambda_2^* = K \) if \( K \leq \Lambda/2 \), solution \((0, \Lambda(1 + a)/2)\) can only become optimal at some sufficiently large value of capacity at which \( \partial \mu_1^*(K)/\partial K < 0 \). As a result, the switching point is, again, unique. \( \square \)
Proof of Lemma 4  For part (i), we provide the proof for the case $\lambda_1^*, \lambda_2^* > 0, \lambda_1^* < (1 - x^*)K, \lambda_2^* < x^* K$. The proofs for the other cases are identical in spirit and thus omitted. Solution $\{(\lambda_1^*, \lambda_2^*): \lambda_1^* \lambda_2^* > 0, \lambda_1^* < (1 - x^*)K, \lambda_2^* < x^* K\}$ satisfies the following stationarity conditions:

\begin{align*}
\partial R(\lambda_1, \lambda_2, x)/\partial \lambda_1 &= 1 - 2\lambda_1/\Lambda + c[\lambda_1/[(1-x)K]] + \lambda_1 c'[\lambda_1/[(1-x)K]]/[(1-x)K] = 0, \\
\partial R(\lambda_1, \lambda_2, x)/\partial \lambda_2 &= 1 + a - 2\lambda_2/\Lambda + c[\lambda_2/(xK)] + \lambda_2 c'[\lambda_2/(xK)]/(xK) = 0, \\
\partial R(\lambda_1, \lambda_2, x)/\partial x &= \lambda_2^2 c'[\lambda_1/[(1-x)K]]/[(1-x)^2 K] - \lambda_2^2 c'[\lambda_2/(xK)]/(x^2 K) = 0.
\end{align*}

We first show that for any allocation $x \in (0, 1)$, there exists at most one solution $(\lambda_1^*(x), \lambda_2^*(x))$ satisfying (12) and (13). (If no such solution exists, the optimal solution must be an extreme point.) In particular, we will show that there exists at most one $\lambda_1^*(x)$ satisfying (13)—the proof that there is at most one $\lambda_1^*(x)$ satisfying (12) is very similar as the two equations differ only by constant $a \geq 0$ once $1 - x$ is replaced by $x$ in (12).

If we let $u \equiv \lambda_2/(xK)$, the LHS of equation (13) becomes $G(u) \equiv 1 + a - 2uxK/\Lambda + c(u) + uc'(u)$. It suffices to show that $G(u) = 0$ cannot have two roots in $(0,1)$. Note that $G(0) = 1 + a + c(0) > 0$ because $c(0) > -1$, and that $\partial G(u)/\partial u = -2xK/\Lambda + 2c'(u) + uc''(u)$. Because $-2xK/\Lambda < 0$, $c''(u) < 0 \Rightarrow uc''(u) < 0$, either $G(u) \downarrow u$ in $(0, 1)$ or $G(u) \uparrow u$ in $(0, 1)$, or $G(u) \uparrow u$ in $(0, u^*)$ and $G(u) \downarrow u$ in $(u^*, 1)$. Thus, $G(u)$ cannot have two roots in $(0, 1)$. Therefore, for any allocation $x \in (0, 1)$, there exists at most one optimal solution to (P3) such that $\lambda_1^*(x) \lambda_2^*(x) > 0$. To complete the proof for part (i), we note that the allocation $x = \lambda_2/(\lambda_1 + \lambda_2)$ satisfies (14) and invoke parts (ii) and (iii) of the lemma, which we show next.

For part (ii), we will show that if $\lambda_1^* \lambda_2^* > 0$, the uniquely optimal allocation satisfies $\lambda_1^* / [(1 - x^*)K] = \lambda_2^* / (x^* K) \Leftrightarrow x^* = \lambda_2^*/(\lambda_1^* + \lambda_2^*)$, i.e., the two capacity segments have the same crowding level at the optimal solution. To that end, let $\lambda_1^* / [(1 - x)K] \equiv u_1$ and $\lambda_2^* / (xK) \equiv u_2$. The objective function in (P3) in terms of $u_1, u_2$ is

\[ R(u_1, u_2) = \lambda_1 + \lambda_2 - (\lambda_1^2 + \lambda_2^2)/\Lambda + \lambda_1 c(u_1) + \lambda_2 c(u_2) + \lambda_2 a. \]

In the revenue function above, fix $\lambda_1, \lambda_2$, where $\lambda_1 \lambda_2 > 0$, and notice that only the term $\lambda_1 c(u_1) + \lambda_2 c(u_2)$ involves allocations $u_1, u_2$. Suppose $u_1 \neq u_2$. Because $c'' < 0$,

\[ [\lambda_1/(\lambda_1 + \lambda_2)] c(u_1) + [\lambda_2/(\lambda_1 + \lambda_2)] c(u_2) < c[u_1 \lambda_1/(\lambda_1 + \lambda_2) + u_2 \lambda_2/(\lambda_1 + \lambda_2)] \]

\[ \Leftrightarrow \lambda_1 c(u_1) + \lambda_2 c(u_2) < \lambda_1 c[u_1 \lambda_1/(\lambda_1 + \lambda_2) + u_2 \lambda_2/(\lambda_1 + \lambda_2)] + \lambda_2 c[u_1 \lambda_1/(\lambda_1 + \lambda_2) + u_2 \lambda_2/(\lambda_1 + \lambda_2)]. \]

Therefore, crowding levels $u_1', u_2'$ such that $u_1' = u_2' = u_1 \lambda_1/(\lambda_1 + \lambda_2) + u_2 \lambda_2/(\lambda_1 + \lambda_2)$ yield strictly higher revenue than crowding levels $u_1, u_2$. As a result, $u_1 = u_2$ at optimality.

Part (iii) follows directly from part 2 if $\lambda_1^* \lambda_2^* > 0$. If $\lambda_1^* = 0$, notice that an allocation $x^* < 1$ cannot satisfy (14), thus it cannot be optimal.

For part (iv), the optimal allocation if $a = 0$ follows directly from the fact that $x^* = \lambda_2^*/(\lambda_1^* + \lambda_2^*)$ and equations (12) and (13). Next we show that $x^*(a) \uparrow a$. Let $u^* \equiv \lambda_1^*/[(1 - x^*)K] = \lambda_2^*/(x^* K)$ so that (12) becomes $F(u^*(x^*), x^*) = 1 - 2u^*(1 - x^*)K/\Lambda + c(x^*) + u^* c'(u^*) = 0$. By the Implicit Function Theorem,

\[ \partial x^*(u^*)/\partial u^* = -\partial F(u^*, x^*)/\partial x^* = -\frac{2u^* K/\Lambda}{2(1 - 1)K/\Lambda + 2c(u^*) + u^* c'(u^*)}. \]
In the last fraction, note that \( u^* > 0 \) and \( x^* < 1 \). We will show that \( 2c'(u^*) + u^*c''(u^*) \leq 0 \) so that \( \partial u^*(x^*)/\partial x^* > 0 \). Using straightforward calculus, \( \partial^2 R(\lambda_1, \lambda_2, x)/\partial x^2|_{\lambda_1 = \lambda_1^*, \lambda_2 = \lambda_2^*, x = x^*} = \partial^2 R(u, x)/\partial x^2|_{u = u^*, x = x^*} = (u^*)^2 K [2c'(u^*) + u^*c''(u^*)]/[x^*(1 - x^*)] \). Because \((u^*, x^*)\) is an optimal solution, \( \partial^2 R(u, x)/\partial x^2|_{u = u^*, x = x^*} \leq 0 \); therefore, \( \partial u^*(x^*)/\partial x^* > 0 \).

Similarly, adding up equations (12) and (13) yields \( H(u^*(a), a) \equiv 2 + a - 2u^* K/\Lambda + 2c(u^*) + 2u^*c'(u^*) = 0 \). By the Implicit Function Theorem,

\[
\frac{\partial u^*(a)}{\partial a} = -\frac{\partial H(u^*(a), a)/\partial a}{\partial H(u^*(a), a)/\partial u^*} = \frac{1}{2[-K/\Lambda + 2c'(u^*) + u^*c''(u^*)]} > 0,
\]

where the last inequality is because \( 2c(u^*) + 2u^*c'(u^*) \leq 0 \). Finally, by the chain rule of differentiation, \( \partial x^*(a)/\partial a = (\partial u^*(a)/\partial a)/\partial u^*(x^*)/\partial x^* > 0 \).

**Proof of Corollary 1** If \( b = 0 \), the objective of (P1) is

\[
R^{P1}(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2 - (\lambda_1^2 + \lambda_2^2)/\Lambda + (\lambda_1 + \lambda_2) c[(\lambda_1 + \lambda_2)/K] + \lambda_2 a,
\]

whereas the objective of (P3) is

\[
R^{P3}(\lambda_1, \lambda_2, x) = \lambda_1 + \lambda_2 - (\lambda_1^2 + \lambda_2^2)/\Lambda + \lambda_1 c[(\lambda_1 + \lambda_2)/((1 - x) K)] + \lambda_2 c[\lambda_1/(x K)] + \lambda_2 a.
\]

Consider now an optimal solution to (P3) \( \{\lambda_1^*, \lambda_2^*, x^* = \lambda_2^*/(\lambda_1^* + \lambda_2^*)\} \) and notice that \((\lambda_1^*, \lambda_2^*)\) is a feasible solution to (P1) and yields the same revenue. Thus, \( \max R^{P1}(\lambda_1, \lambda_2) \geq \max R^{P3}(\lambda_1, \lambda_2, x) \). Likewise, consider an optimal solution to (P1) \( \{\xi_1^*, \xi_2^*\} \) and notice that solution \( \{\xi_1^*, \xi_2^*, x^* = \xi_2^*/(\xi_1^* + \xi_2^*)\} \) is a feasible solution to (P3) and yields the same revenue. Thus, \( \max R^{P1}(\lambda_1, \lambda_2) \leq \max R^{P3}(\lambda_1, \lambda_2, x) \). The last condition along with \( \max R^{P3}(\lambda_1, \lambda_2, x) \) jointly imply that \( \max R^{P1}(\lambda_1, \lambda_2) = \max R^{P3}(\lambda_1, \lambda_2, x) \). Because both (P1) and (P3) have unique optimal solutions, they must have the same optimal solution.

**Proof of Theorem 1** Note that if \( b = 0 \), the optimal solutions and the revenues are the same with or without capacity allocation, as Corollary 1 suggests. Further, by the Envelope Theorem, when classes do not interact, revenue is not a function of \( b \). On the other hand, when classes interact, \( \partial [\max R(\lambda_1, \lambda_2)]/\partial b = \lambda_1^* \lambda_2^*/(\lambda_1^* + \lambda_2^*) \geq 0 \). Hence, the result.

**Proof of Theorems 2** In the entire proof we make (implicit) use of the fact that if \( b_1 = b_2 = 0 \), the optimal solutions and the revenues are the same with and without capacity allocation. Also, note that if \( \lambda_1^* = 0 \) in some region of the \( b_1 \times b_2 \) space, revenue is invariant of \( b_1, b_2 \) in that region. Thus, throughout this proof, we focus on the case in which \( \lambda_1^* \lambda_2^* > 0 \) in problem (P2) unless we note otherwise. Consider now stationarity conditions (10) and (11). Because \( \lambda_1^* \lambda_2^* > 0 \), \( \mu_2 = \mu_3 = 0 \); subtracting (10) from (11) yields the optimal Lagrange multiplier \( g^* = \lambda_2 - \lambda_1 - a/[b/(\lambda_1^* + \lambda_2^*) + 2/\Lambda] \). To study the behavior of the optimal revenue function wrt \( b_1, b_2 \), we need the term in the optimal Lagrange function that depends on \( b_1, b_2 \). The relevant term is \( L(\lambda_1^*, \lambda_2^*) \equiv b \lambda_1^* \lambda_2^*/(\lambda_1^* + \lambda_2^*) + g^* \{b_1/(\lambda_1^* + \lambda_2^*) + 1/\Lambda \lambda_2^* - [b_2/(\lambda_1^* + \lambda_2^*) + 1/\Lambda] \lambda_1^* - a\} \).

Parts (i) and (iv) of the theorems follow from the fact that \( R(\lambda_1^*, \lambda_2^*) \uparrow b_1, R(\lambda_1^*, \lambda_2^*) \uparrow b_2 \), which we showed in the proof of part (i) of Proposition 2.

Next we show parts (ii) of both theorems, which require \( b_2 > 0 \). If \( b_1 \leq a - K/\Lambda \), we know from Lemma 2 that without capacity allocation, an exclusive system is optimal. Because the revenue of an exclusive system...
can be replicated by allocating capacity $x = 1$, allocating capacity can only improve revenue. Hence, part (ii)-a of both theorems. To show parts (ii)-b and (ii)-c of both theorems, it suffices to show the following: 1) $R(\lambda_1^*, \lambda_2^*) \uparrow b_1$, $R(\lambda_1^*, \lambda_2^*) \uparrow b_2$; 2) If $b = 0$ and $b_2 > b_1$, it is optimal to allocate capacity. We have already shown (1) and to show (2), suppose that $b_1 = -b_2$. In that case, constraint (4) implies $\lambda_2^* - \lambda_1^* = (a + b_2)\Lambda$; thus, $g^* = \lambda_2^* - \lambda_1^* - a/(2/\Lambda) = (a/2 + b_2)\Lambda > 0$. Now recall that if $b_1 = b_2 = 0$, revenues are the same with and without capacity allocation, and note that for fixed $b$, $\partial L/\partial b_2 - \partial L/\partial b_1 = -g^* < 0$. Therefore, if $b = 0$ and $b_2 > 0 > b_1$, it is optimal to allocate capacity.

Next we show part (iii) of Theorem 2, which requires $b_1 > 0 > b_2$. If $b_2 \leq -K/\Lambda$, or $b_1 \leq a - K/\Lambda$ and $b_2 > -K/\Lambda$, we know from Lemma 2 that without capacity allocation, an exclusive system is optimal. In that case, as we argued earlier, allocating capacity yields the same or higher revenue. Hence, part (iii)-a. Suppose now that $b_1 > a - K/\Lambda$ and $b_2 > -K/\Lambda$. If $\lambda_1^* = 0$ in problem (P2), the proof for part (iii)-a of the theorem applies.

To prove part (iii)-c of Theorem 2, we will show that the revenue from not allocating capacity strictly increases in $b_1 - b_2$ if $b_1 = -b_2$ and $\lambda_1^* > 0$ in problem (P2). Recall that if $b_1 = -b_2$, constraint (4) implies $\lambda_2^* - \lambda_1^* = (a + b_2)\Lambda \Rightarrow g^* = \lambda_2^* - \lambda_1^* - a/(2/\Lambda) = (a/2 + b_2)\Lambda$; thus, $g^* \geq 0$ if $b_2 \geq -a/2$. Now recall that if $b_1 = b_2 = 0$, revenues are the same with and without capacity allocation, and note that for fixed $b$, $\partial L/\partial b_1 - \partial L/\partial b_2 = g^* \leq 0$, where the last inequality is strict if $b_2 > -a/2$. Therefore, if $b = 0$, $a/2 \geq b_1 > a - K/\Lambda$, $b_2 \geq -a/2$ and $\lambda_1^* > 0$ in problem (P2), it is strictly optimal to not allocate capacity. ∎