Discount Rates and Tax

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Abstract

This note summarises the relationships between values, rates of return and betas that depend on taxes. It extends the standard analysis to include the effect of risky debt. It brings together a variety of results that are often misunderstood or misinterpreted. Both the WACC and APV approaches are presented for a generalised tax system that encompasses both classical and imputation systems. It shows how basic assumptions about the tax treatment of the ‘representative’ investor, the firm’s dividend policy, the firm’s leverage policy and the riskiness of the tax savings from interest give rise to particular expression for leveraged and unleveraged betas and discount rates. Results for the Miller-Modigliani and Miles-Ezzell assumptions are summarised in detail and presented in a simple table.

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1 Introduction

A common source of confusion and disagreement in corporate finance is the effect of taxes on valuation and rates of return. There are alternative approaches to the treatment of tax in the cost of capital, the value of the tax saving from debt, switching post-tax to pre-tax returns, the correct version of the capital asset pricing model to use in the presence of taxes, the impact of an imputation tax, and many other tax-related issues.

Some of these differences represent substantive variation of assumptions, such as different assumptions about the tax treatment of the investors that are important in setting the share price of a company. Others represent different views on how the future leverage policy of the company will be determined. In other cases, however, differences represent inconsistencies and confusion.

The purpose of this note is to show how all relationships that are commonly used in this area stem from a few basic assumptions. Differences in these basic assumptions generate different relationships between leveraged and unleveraged values, and leveraged and unleveraged discount rates. A consistent approach involves understanding the basic assumptions one wants to use and then using the relationships and estimation procedures that are consistent with those assumptions.

2 Basic Assumptions

All relationships between values, discount rates and betas that are affected by leverage and tax start from some basic assumptions. These concern:

- The tax treatment of the ‘representative’ investor
- The firm’s dividend policy
- The firm’s leverage policy
- The riskiness of the tax savings from interest
- The cost of financial distress
2.1 Tax rates and dividend policy

The notion of a ‘representative investor’ is a common one in finance. It means the investor (or weighted average of investors) who is important in pricing the company’s shares at the margin. As such it is an almost tautological concept and, in practice, the identification of the tax rate of the representative investor is very difficult. It is discussed further in section 5 below where the impact of dividend policy, which affects this tax rate, is also analyzed.

Two extreme assumptions about the representative investor are the ‘Modigliani and Miller’ assumption, that this investor pays no taxes, and the ‘Miller’ assumption, that this investor pays tax on interest that exceeds the tax rate on equity by an amount equal to the corporate tax rate. We deal with both of these, as well as intermediate assumptions. They are discussed in section 5.

2.2 Leverage policy

The two main approaches to leverage policy are the Modigliani and Miller (1963) (MM) and the Miles-Ezzell (1980) (ME) approaches. The difference is that ME assume that the amount of debt is adjusted to maintain a fixed market value leverage ratio, whereas MM assume that the amount of debt in each future period is set initially and not revised in light of subsequent developments. In section 4 we use the MM assumptions. In section 7 we show how much difference the ME assumptions make.

2.3 Riskiness of the tax saving

A common assumption about the riskiness of tax saving from interest is that it is equal to the riskiness of the firm’s debt. This need not, however, be the case. For instance, under the ME assumptions the riskiness of the tax saving is closer to the risk of the assets of the firm. Kaplan and Ruback (1995) make this assumption, for highly levered structures, without assuming the ME leverage policy. The impact of their assumption is discussed in section 10.
2.4 The cost of financial distress

The formulae we give for the effect of leverage on discount rates and values ignore the costs of financial distress. To give the overall effect of leverage on value, the impact of expected future distress costs must be added to the tax effects. We do not discuss how to do this. A good discussion can be found in Brealey and Myers (2003).

3 Valuing The Leveraged Firm

3.1 The General Case

In general, the value of the leveraged firm including the tax effect of debt is the unleveraged value \( V_A \) plus the present value of the tax savings from debt. As long as there will always be enough taxable income to use all the interest charges to save tax, we have:

\[
V_L = V_A + \sum_{t=1}^{\infty} \frac{E(TS_t I_t) / (1 + R_{TS})^t}{1 + R_{TS}}
\]

where \( E(\cdot) \) is the expectations operator, \( I_t \) is the interest payment at date \( t \), \( TS_t \) is the tax that will be saved at date \( t \) per dollar of interest charges, and \( R_{TS} \) is the discount rate appropriate to the tax saving. In order to use this equation in practice, we must estimate three things: (i) the unleveraged value, (ii) the discount rate for the tax shield, and (iii) the expected net tax saving from interest deductions in each future period.

The value relationship given by (1) provides us with a framework for computing the tax value of leverage. We can put into this expression whatever future plan for leverage we predict. Combined with assumptions about the costs of financial distress it also tells us something about optimal capital structure. We can also use it to value projects within the firm, taking into account the incremental tax shield generated by a particular project.

However, in practice, there are several complexities that arise in implementation of the formula. The first is the definition of the appropriate rate of tax saving. Two issues arise here. One is the impact of personal taxes. This is discussed extensively in section 5. The other is the rate at which corporate tax is saved.

\(^1\) The assumption throughout is that capital markets are complete, so that any cash flow stream has a well-defined value.
This should be the incremental rate at which the tax deduction arising from interest saves tax. It will not necessarily be equal to the average rate of tax paid by the company. A second complication is that future tax savings are uncertain. Future statutory tax rates and tax systems are not known, and could vary. Also, the tax position of the company may change. For instance, in some circumstances it may not have enough taxable income to pay tax. In such a case, the future tax payment is like a call option on the taxable income of the company. This raises complex valuation issues, that are beyond the scope of this paper.

As a first approximation, it is common to make simple assumptions about future leverage and tax rates. The most common assumptions about leverage are the MM assumption, that the future amount of debt will remain constant, and the ME assumption that the future leverage ratio will remain constant. The benefit of these assumptions is that they lead to relatively simple expressions for discount rates that include the tax benefit of borrowing, making it easy to put the tax effect of borrowing into a valuation. We now derive these expressions for the MM assumptions. In section 7 we show similar results for the ME assumptions. Assumptions about tax rates are discussed in section 5.

4 Generalized MM Assumptions

In this section we derive the relationships between value, discount rates and betas for leveraged and unleveraged firms using a generalized version of the MM assumptions. We generalize their assumptions by including personal as well as corporate rates.

4.1 Assumptions and Notation

Cash Flows (MM)

- The firm generates a risky perpetuity of an expected amount \( C \), which is taxable. After corporate tax this is equal to \( C(1 - T_C) \).

Financing

- The dollar amount of debt is a constant amount of perpetual debt, \( D \), at a fixed interest rate, \( R_D \).
- The value of equity is \( E \).
Tax

- The corporate tax rate is $T_C$.
- Corporate interest payments are tax deductible.
- The tax rate on equity flows to the representative investor is $T_{PE}$.
- The representative investor is taxed at $T_{PD}$ on debt flows.

Capital market rates and prices

- $R_F$ is the risk-free rate.
- $R_A$ is the required return on equity after corporate tax if the firm has no leverage.
- $R_E$ is the required return on equity in the leveraged firm after corporate tax.
- $V_A$ is the value of the unleveraged firm.
- $V_L = D + E$ is the total value of the debt and equity of the leveraged firm (sometimes called ‘enterprise value’).
- $I = R_D D$ is the total expected interest charge.

4.2 The value of the unleveraged firm

Suppose that the firm is unleveraged. Before tax it generates a perpetuity of $C$. Let $c_E$ be the after-tax cash flow that the investor receives per dollar of pre-tax corporate cash flow. Then:

$$c_E = (1 - T_C)(1 - T_{PE})$$

---

2 The existence of a ‘representative investor’ means that we can value all cash flows as though they are received by this investor. This is a non-trivial assumption. The interested reader can find an excellent discussion in Duffie (1992).

3 Care must be taken to distinguish between promised debt payments and expected debt payments. Expected payments are promised payments adjusted for the probability of default. Thus the common practice of setting the expected return on debt equal to the promised yield assumes that there is zero probability of default. See Cooper and Davydenko (2003).
The investor’s after tax required return is $R_A(1 - T_{PE})$ after corporate and personal taxes. So the value of the unleveraged firm is the investor’s after-tax cash flow discounted at the after-tax required rate of return for a cash flow with this level of risk:

$$V_A = \frac{C(1 - T_C)(1 - T_{PE})}{R_A(1 - T_{PE})} = \frac{C(1 - T_C)}{R_A}.$$  \hspace{1cm} (3)

This illustrates the general principle when dealing with the impact of taxes: when in doubt discount after-tax cash flows to the representative investor at the representative investor’s after-tax required return for that level of risk.

### 4.3 The value of the leveraged firm

As we are interested only in the tax impact of leverage, we assume that the firm pursues the same operating policy regardless of its amount of leverage.\(^4\) So the pre-tax cash flow, $C$, is the same for the leveraged firm as for the unleveraged firm. Leverage simply takes cash flow that would be paid to equity holders in the unleveraged firm and pays it out to debt.

The net tax advantage to debt is, therefore, the value of the difference between the after tax cash flow, $c_D$, that an investor receives when a dollar of pre-tax corporate cash flow is paid out as interest and the after tax cash flow received when a dollar of corporate pre-tax cash flow is allocated as a return to equity, $c_E$. It is straightforward that, due to the corporate-level tax deductibility of interest payments, only personal tax is paid on cash flows distributed as debt:

$$c_D = 1 - T_{PD}$$  \hspace{1cm} (4)

Subtracting (2) from (4), the net tax advantage to debt per dollar of pre-tax earnings paid as interest rather than to equity is:

$$T_S = (1 - T_{PD}) - (1 - T_C)(1 - T_{PE})$$  \hspace{1cm} (5)

\(^4\)A more general treatment of the effect of leverage would include costs of financial distress and agency effects.
The leveraged firm generates a total equity flow equal to \([C - I](1 - T_C)\] and a total debt flow equal to 
\(I\). After investor tax the total of these flows is:

\[
(C - I)(1 - T_C)(1 - T_{PE}) + I(1 - T_{PD}) = C(1 - T_C)(1 - T_{PE}) + IT_S
\]  

(6)

The first term is the cash flow received by the equity holders in the unleveraged firm. The second is the extra after-tax flow received by the aggregate of all debt and equity holders in the leveraged firm. The net cashflow to the aggregate of all investors in the leveraged firm is \(IT_S\) larger than the net cash flow to the aggregate of all debt and equity investors in an equivalent unleveraged firm.

For valuation purposes, both of the flows in (6) can be considered as going to the same investor (the representative investor), so we can get the value of the leveraged firm by considering the value of the total flow. The first part of this flow is identical to the after-tax flow from an unleveraged firm, and so has the same value, \(V_A\). The second part is the after-tax flow from the corporate tax saving net of the personal tax effect resulting from using debt rather than equity financing.

In general, the expected tax saving from debt should be discounted at a rate, \(R_{TS}\), that reflects the risk of the tax saving, so that the value of the tax shield is:

\[
V_{TS} = \frac{R_D DT_S}{R_{TS}}. 
\]  

(7)

An important assumption of MM with risky debt is that the tax saving from debt has the same risk as the debt. As a consequence, it should be discounted at the investor’s after-tax discount rate for equity flows that have the same risk as debt. This must be equal to the after tax return on debt itself: \(R_{TS} = R_D(1 - T_{PD})\). This makes the value of the tax saving from debt:

\[
V_{TS} = \frac{R_D D[(1 - T_{PD}) - (1 - T_C)(1 - T_{PE})]}{R_D(1 - T_{PD})}. 
\]  

(8)

So the value of the leveraged firm is:

\[
V_L = V_A + D \left[1 - [(1 - T_C)(1 - T_{PE})/(1 - T_{PD})]\right]. 
\]  

(9)
We define a variable $T^*$ that represents the value increase for an extra dollar of debt rather than equity financing, in the MM world with personal taxes, by:

$$T^* = \frac{T_S}{(1 - T_{PD})}$$  \hspace{1cm} (10)

This also satisfies:

$$(1 - T^*) = (1 - T_C)(1 - T_{PE})/(1 - T_{PD}).$$  \hspace{1cm} (11)

which is an expression that we will use extensively. Then (9) gives:

$$V_L = V_A + T^* D.$$  \hspace{1cm} (12)

The value of the firm rises with leverage by $T^*$ multiplied by the amount of debt. This is the fundamental value relationship in the extended MM model. The implication is that when

$$(1 - T_{PD}) > (1 - T_C)(1 - T_{PE})$$  \hspace{1cm} (13)

then $T^* > 0$ and there is a tax advantage to debt, in the sense that the value of the firm rises as more debt is taken on. When the inequality is reversed there is an advantage to equity.\(^5\)

5 Determinants of the tax rate on equity and the net tax advantage to debt.

In most countries, corporations can deduct interest payments from their earnings before taxes, giving rise to an apparent tax advantage to debt financing relative to equity financing. In general, the value of a leveraged firm is the value of the firm if financed entirely with equity (the ‘all equity firm’) plus the value of the tax shield arising from the tax deductibility of interest. Valuing the tax shield requires knowledge of the net tax saving to debt financing relative to equity financing. In practice, this will often involve subjective judgement.

\(^5\)Note that this is true whatever the discount rate for the tax saving.
However, it is important to understand how to use actual tax rates to make reasonable assumptions about the net tax saving to debt. This is the main issue addressed in this section.

5.1 Taxation of shareholders

The tax rate on equity, $T_{PE}$, is in fact a combination of various elements of the taxation of shareholders:

- The dividend payout ratio $\alpha$. This is the fraction of the return on equity that takes the form of dividends.\(^6\)

- $T_{PEC}$ the tax on equity capital gains, and $T_{PED}$ the tax on gross dividends.

- The rate of imputation tax (if relevant) $T_I$.

5.2 Imputation tax

The standard papers on capital structure and tax all relate to the US tax system. This is a 'classical' tax system, where dividend payments are fully taxed. In many other countries there is a further complication: the imputation tax. This system was considered, but eventually not implemented, by the US in 2004. Under an imputation system, a part of the tax payment by a company is imputed to be paid on behalf of shareholders. The way this works is typically in conjunction with dividend payments. As an illustration, suppose a company makes a dividend payment of $Div$. Under a classical tax system, the investor’s after-tax dividend would be $Div(1 - T_{PED})$. Under the imputation tax system, however, the tax authority operates with the concept of a gross dividend, defined as the dividend payment grossed up by the imputation tax, that is $Div/(1 - T_I)$. While the investor is liable for tax on the gross dividend, he is imputed to already have paid the rate of imputation tax on this dividend. The investor’s after tax cash flow is, therefore,

\[
Div \frac{(1 - T_{PED})}{(1 - T_I)}
\]

and the net payment of tax by the investor is

\(^6\)This is different from the normal payout ratio, which is the ratio of dividends to earnings.
\[ D_{\text{iv}} \frac{(T_{PED} - T_I)}{(1 - T_I)}. \]

In the case that \( T_{PED} < T_I \), the investor should receive money from the tax authority. Tax authorities vary as to whether they repay this amount.

An imputation tax system enhances the tax advantage to dividend payments and reduces the net tax advantage to debt. This is demonstrated below.\(^7\) In what follows, the results for a classical tax system can be obtained by setting \( T_I = 0 \).

### 5.3 The net tax advantage to debt under imputation

Per dollar of pre-tax cash flow paid as a dividend, the investor collects after tax:

\[ (1 - T_C) \frac{(1 - T_{PED})}{(1 - T_I)}. \]  \hspace{1cm} (14)

Retained earnings give rise to a capital gain. The after tax value to an investor per dollar of retained earnings is, therefore:

\[ (1 - T_C)(1 - T_{PEC}). \]  \hspace{1cm} (15)

Keeping in mind that \( \alpha \) is the payout ratio, we have the investor’s after tax cash flow per dollar of pre-tax corporate cash flow:

\[ c_E = \alpha(1 - T_C) \left( \frac{1 - T_{PED}}{1 - T_I} \right) + (1 - \alpha)(1 - T_C)(1 - T_{PEC}). \]  \hspace{1cm} (16)

We can define the average tax rate on equity by \( T_{PE} \) such that

\[ (1 - T_{PE}) = \alpha \left( \frac{1 - T_{PED}}{1 - T_I} \right) + (1 - \alpha)(1 - T_{PEC}). \]  \hspace{1cm} (17)

Note that \( T_{PE} \), the average tax rate on equity returns, depends on the payout ratio, \( \alpha \).

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\(^{7}\)One other consequence of an imputation system is that cash flows that are post-tax to the corporation are not the same as pre-tax cash flows to the investor, since the investor recaptures part of the corporate tax through the imputation system. So we must be careful to distinguish between post-corporate-tax returns and pre-investor tax returns.
Then equation (16) simplifies to

\[ c_E = (1 - T_C)(1 - T_{PE}) \]  

(18)

where \( T_{PE} \) is defined by (17).

The net tax saving to debt is given by:

\[ T_S = (1 - T_{PD}) - \alpha \left( \frac{1 - T_{PED}}{1 - T_I} \right) + (1 - \alpha)(1 - T_{PEC}) \left( 1 - T_C \right) = (1 - T_{PD}) - (1 - T_{PE})(1 - T_C) \]  

(19)

which looks just like (5), except that \( T_{PE} \) is now defined by payout policy and various tax rates, as given by (17). So, depending on the values of \( T_{PD}, T_C, T_{PEC}, \alpha, \) and \( T_{PED} \), the tax saving, \( T_S \) can be positive, negative or zero.

\( T_{PEC} \) and \( T_{PED} \) usually differ, in part because of an investor’s ability to shield capital gains by selling losers, or defer capital gains by by not selling winners. However, in most cases, \( T_{PED} = T_{PD} \), that is income from dividends and interest are taxed at the same rate, apart from the effect of imputation. Applying this assumption and rearranging (19) gives:

\[ T_S = \frac{(T_C - T_I)(1 - T_{PD})}{1 - T_I} - (1 - \alpha)(1 - T_C) \left[ \frac{1 - T_{PD}}{1 - T_I} - (1 - T_{PEC}) \right] \]  

(20)

The corresponding expression for \( T^* \) is:

\[ T^* = \frac{(T_C - T_I)}{(1 - T_I)} - (1 - \alpha)(1 - T_C) \left[ \frac{1}{1 - T_I} - \left( \frac{1 - T_{PEC}}{1 - T_{PD}} \right) \right] \]  

(21)

This reveals the effect of imputation on the tax saving from debt. The first term shows that imputation effectively reduces the corporate tax rate, as it protects investors from further tax even if the distribution is paid as dividends rather than interest. The second term shows that this depends on the payout ratio, because the imputation credit is attached to dividends. It also depends on the level of the imputation credit, in particular the degree to which it offsets the tax difference between dividends and capital gains taxes.

5.4 The payout ratio

The payout ratio is usually assumed to be the same for the leveraged as for the unleveraged firm. A more reasonable assumption, however, might be that the company retains the same total amount of cash under
different leverage strategies, in order to pursue the same operating strategy. In that case, the payout ratio of a leveraged firm would be lower than that of an identical unleveraged firm.

This would complicate the analysis, as the equity tax rate for the leveraged firm would be different from that of the unleveraged firm. In what follows, for ease of notation, we will drop the dependency of $T_{PE}$ on the payout ratio. But the reader should bear in mind that $T_{PE}$ is a function of the payout ratio as well as on tax rates on capital gains and dividends.

### 5.5 Standard assumptions about the size of the tax saving from debt

The key variables that define the tax saving from debt are $T^*$ and $T_S$, given by (11) and (5). In principle, these could take any value between zero and $T_C$. In practice, there are three assumptions that are commonly used. The first are those typically seen in the US and other classical tax systems:

**Classical Tax System ($T_I = 0$)**

- Original MM: $T_{PE} = 0, T_{PD} = 0, T_S = T_C, T^* = T_C, V_L = V_A + T_C D$
- ‘Miller’: $T_{PE} = 0, T_{PD} = T_C, T_S = 0, T^* = 0, V_L = V_A$

The original MM model assumes that the representative investor pays no tax, so the value of the corporate tax shield reflects the full corporate tax rate ($T^* = T_C$). This is the version often used in the US. In contrast, in the Miller (1977) model the tax advantage to the corporation is fully offset by a tax disadvantage to debt for the representative investor, so there is no net tax advantage to borrowing ($T^* = 0$).

This third assumption is commonly used in countries with imputation taxes:

**Imputation ($T_I > 0$).**

- $T_S = (1 - T_{PD}) - (1 - T_{PE})(1 - T_C)$, where $T_{PE}$ is given by (17), $T^* = 1 - (1 - T_C)(1 - T_{PE})/(1 - T_{PD})$, $V_L = V_A + T^* D$.

An interesting special case that is often used is that $T_{P \cdot D} = T_{PD}$ and the payout ratio, $\alpha$, is 100%. In that case, which is effectively the imputation tax version of the MM assumptions, the tax saving from debt is given by:

$$T^* = (T_C - T_I)/(1 - T_I)$$ (22)
In this case, the net tax saving is lowered by the effect of imputation. As a consequence, some people prefer to think of the imputation system as reducing the effective corporate tax rate to $T_C$, where $(1 - T_C) = (1 - T_C)/(1 - T_I)$. With this definition of an adjusted corporate tax rate, the standard MM formulae can be used.

In contrast, Miller’s argument that justifies the assumption that $T^* = 0$ is not affected by an imputation system. In Miller’s original setting the representative investor has an equilibrium tax rate equal to $T_C$ on debt and zero on equity. This tax discrimination in favour of the investor receiving equity payments exactly offsets the tax discrimination in favour of the company making debt payments. In the imputation setting, the representative investor that satisfies the Miller equilibrium is any investor whose tax status, $(T_{PD}, T_{PED}, T_{PEC})$, satisfies $T_S = 0$ where $T_S$ is given by (19). For example, if $\alpha = 1$ then:

$$T_{PE} = 0 \text{ and } (1 - T_{PD}) = (1 - T_C)/(1 - T_I)$$

(23)

gives the Miller result, in the sense that $T_S = 0$.

5.6 Empirical estimation of the tax saving from debt

The issue of which assumption about the net tax benefit of debt is correct is an empirical one. Empirical studies of the actual value of $T^*$ for the US have failed to reach any definitive conclusion on this issue. Fama and French (1998) fail to find any increase in firm value for debt tax savings, implying a value of $T^*$ of zero. In contrast, Kemsley and Nissim (2002) find that $T^*$ is 40%, similar to the corporate tax rate. It is fair to say that the value of $T^*$ remains an open question. Graham (2000) estimates a value of $T^*$ that is intermediate between these two extremes based not on personal taxes, but on different corporate tax positions. It might seem that uncertainty about such an important valuation parameter should have been resolved by now. The reason that it has not is that it is extremely difficult empirically to distinguish between the impact of leverage on value and the impact of other things with which leverage is associated, such as profitability.
6 Relationships between returns under the MM Assumptions of a Constant Debt Level

The impact of leverage on value means that it also affects rates of return. In this section we focus on the effect of leverage on relationships between expected rates of return on assets, equity and debt. These are key inputs to valuations, so knowing how to adjust them for leverage is important. A summary of the results in this and other sections is given in Table 3. Table 4 shows the same results under the more familiar assumption that there are no personal taxes, so that \( T^* = T_C \).

When estimating discount rates, a common approach in practice is to start from the cost of equity and compute a weighted average cost of capital (WACC), defined by:

\[
WACC = \frac{E}{(D+E)}R_E + \frac{D}{(D+E)}R_D(1-T_C) \tag{24}
\]

Given the current leverage of the firm, the WACC is intended to estimate the discount rate that may be used to discount operating cash flows after tax to give a value that includes the tax benefit of borrowing. It is the correct rate for this purpose in only two circumstances. One is the MM assumption of a constant debt level combined with an expected operating cash flow that is a flat perpetuity. Only in these restrictive circumstances is the WACC expected to be the same over time if the MM assumptions are used. The other, more general, assumption that makes the WACC the correct discount rate is when leverage will be maintained at a constant proportion of value in all future periods. This section discusses the former case, section 7 discusses the latter case, and section 8 the general case.

In a taxfree world, or in a world where \( T_S = 0 \), there is no tax benefit to borrowing, so the WACC is equal to the discount rate for an all-equity firm, \( R_A \). More generally, however, the WACC is not identical to \( R_A \) because WACC takes the interest tax shield into account, while \( R_A \) does not. Sometimes we want a discount rate that does not include the tax benefit of borrowing, so we need to know how to go from the WACC to the unleveraged (all-equity) rate. Sometimes we want to get a rate that reflects a different amount of leverage, \( R_L \). Sometimes we also want to know how the cost of equity will respond to leverage, so that we

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\* An alternative is to use asset betas, which are discussed below.
can use an appropriate rate to discount a stream of equity cash flows, and we need to know the relationship between the leveraged discount rate for equity, $R_E$ and $R_A$.\footnote{Although it is normal to perform valuations using cash flows that are post-tax to the company, some companies and regulators are interested in using pre-tax required returns to set targets. Thus they are interested in computing the pre-tax return that is equivalent to a particular post-tax return.}

Thus the two relationships that we are interested in are:

- The relationship between the all-equity discount rate, $R_A$, and the WACC (or $R_L$).

- The relationship between the all-equity rate of return, $R_A$, and the leveraged equity rate of return, $R_E$.

### 6.1 The relationship between WACC and $R_A$

To derive relationships among different rates of return we use the value relationship (12). Appendix B shows that:

\[
R_A = \frac{WACC}{[1 - T^* D/V_L]}. \tag{25}
\]

Since we usually start by computing the WACC, the point of relationship (25) is that it enables us to unleverage the WACC to calculate $R_A$. This can then be leveraged up to a WACC that corresponds to a different debt ratio if we want to. This ability to leverage up and down the required return is important when we consider different leverage strategies for a company or when we consider projects whose incremental contribution to debt capacity is different to the leverage reflected in a company’s WACC.

### 6.2 The relationship between $R_E$ and $R_A$

Appendix B also shows that the relationship between the cost of equity and $R_A$ is given by:

In general the way to switch from post-tax to pre-tax required returns is to compute the pre-tax economic return that is required to give the appropriate level of post-tax return. This will depend upon asset profiles and tax accounting rules as discussed in Dimson and Staunton (1996). The relationship will not, in general, be close to any simple calculation based on crude simplifying assumptions.
\[ R_E = R_A + (D/E)[R_A(1 - T^*) - R_D(1 - T_C)]. \] (26)

This can be used to compute the cost of equity that corresponds to any given leverage, starting from the unleveraged cost of equity.

### 6.3 The relationship between \( R_A, R_E, R_D \) and \( R_F \) (the CAPM)

Appendix B shows that the version of the CAPM that is consistent with the assumptions about tax that determine \( T^* \) is:

\[
R_E = \frac{R_F(1 - T_C)}{(1 - T^*)} + \beta_E P = \frac{R_F(1 - T_{PD})}{(1 - T_{PE})} + \beta_E P
\] (27)

where \( \beta_E \) is the beta of the equity and where:

\[
P = R_M - R_F \frac{(1 - T_{PD})}{(1 - T_{PE})} = R_M - R_F \frac{(1 - T_C)}{(1 - T^*)}
\] (28)

This is the market risk premium after personal taxes grossed up by \( (1 - T_{PE}) \). \( R_M \) and \( R_F \) are measured in the standard way, using returns before investor taxes. Betas are also measured in the standard way, using pre-tax returns.

Note that only if \( T^* = T_C \) is the standard version of the CAPM, with an intercept equal to \( R_F \), valid. In particular, this means that the assumption that \( T^* = T_C \) corresponds to the normal CAPM, whereas the ‘Miller’ assumption that \( T^* = 0 \) corresponds to a CAPM where the intercept is \( R_F(1 - T_C) \). A similar effect can be seen in the formula for the required return on assets:

\[
R_A = \frac{R_F(1 - T_C)}{(1 - T^*)} + \beta_A P = \frac{R_F(1 - T_{PD})}{(1 - T_{PE})} + \beta_A P
\] (29)

The required return on debt follows a different version of the CAPM, because the tax treatment of debt and equity differ in all cases other than the standard MM case:

\[ R_D = R_F + \beta_D P \] (30)
Regardless of the assumption about taxes, the pre-tax CAPM holds for debt, because all debt is taxed in the same way.

### 6.4 Relationships between Betas

Under the MM assumptions, that the riskiness of the tax shield equals the riskiness of the debt that generates it, and debt capacity is constant, Appendix B shows that (26) can be rearranged to give:

$$\beta_A = \frac{E}{V_L - T^*D}\beta_E + \frac{D}{V_L - T^*D}\beta_D(1 - T_C) \tag{31}$$

From this relationship we can derive other relationships among betas given in Table 3 for the extended MM model.\(^{10}\)

### 7 An Alternative Assumption: Miles-Ezzell (ME)

All the above results have been derived using the generalized MM assumptions. These are restrictive, in that they require that all expected cash flow streams are level perpetuities and a fixed amount of debt. A more realistic alternative is the ME assumption of constant market value leverage. ME assume that the debt will be adjusted in each future period to be a constant proportion of the total market value of the firm. With this assumption, any pattern of future cash flows can be accommodated. Its importance is that, under the ME assumptions, the WACC formula (24) gives the correct discount rate to calculate the leveraged value of the firm, regardless of the pattern of future cash flows.

#### 7.1 The Miles-Ezzell Formula

The ME assumptions lead to slightly different formulas to the MM assumptions. We derive the ME formulas in Appendix C. The standard version of the ME formula looks slightly complicated, but the complication comes from the fact that ME assume that the leverage ratio is adjusted only once a year. If leverage is

\(T_C\) is replaced by \(T^*\).
constantly adjusted, we get a simpler formula:

\[ R_L = R_A - LT^* R_D \frac{(1 - T_C)}{(1 - T^*)} = R_A - LR_D \frac{T_S}{(1 - T^*)} \]  

(32)

In the case where \( T^* = T_C \), this simplifies to:

\[ R_L = R_A - LT_C R_D \]  

(33)

Both this and the more complex version of the ME formula are approximations. It is not clear which is more accurate, and it does not make a large difference, so we use the simpler version. This is also the version that underlies the standard formula for asset betas. A summary of useful relationships using this version of the ME model is given in the final column of Table 3.\textsuperscript{11}

### 7.2 Comparison of MM and ME assumptions

The relationship between the MM and ME formulae can be seen by considering a firm that generates a set of cash flows with a constant growth rate. Ignoring personal taxes, the leveraged value of the company using the ME formula is:

\[ V_L = C(1 - T_C)/(R_L - g) \]

(34)

\[ = C(1 - T_C)/(R_A - LR_D T_C - g) \]

If the company had no leverage its value would be:

\[ V_A = C(1 - T_C)/(R_A - g) \]  

(35)

The value of the tax saving is the difference between these values:

\[ V_{ME}^{TS} = C(1 - T_C)/(R_A - LR_D T_C - g) - C(1 - T_C)/(R_A - g) \]

(36)

\[ = DR_D T_C/(R_A - g) \]

\textsuperscript{11}Simpler-looking versions of these formulae can be derived by substituting \( R_{FE} = \frac{R_{F}(1 - T_C)}{(1 - T^*)} \), as in Taggart (1991). However, we prefer to leave the dependence on \( T^* \) in the formulae explicit.
Thus, the value of the tax saving is the value of a growing perpetuity starting at $DR_D T_C$, growing at $g$, with risk the same as the asset.

We can contrast this with the MM assumptions by setting $g$ equal to zero. Then the value is:

$$V_{TS}^{ME} = DTCR_D / RA$$ (37)

Whereas the value of the tax savings according to MM is:

$$V_{TS}^{MM} = DTCR_D / RD = DTC$$ (38)

The difference is that the tax saving in ME is discounted at the required return on assets, whereas, in MM it is discounted at the required return on debt. So MM does not represent simply the ME assumption with zero growth. It is a completely different financing strategy. Even with cash flows that are expected to be perpetuities, the MM and ME assumptions differ. MM assume that the amount of debt will not change, regardless of whether the actual outcome of the risky perpetuity is higher or lower than its expected value, whereas ME assume that it will rise and fall in line with the expected cash flow.

### 8 Adjusted present value (APV)

If neither the MM nor the ME assumptions about future cash flows and capital structure are fulfilled, then the WACC cannot be used to value the firm. However, the general formula (1) may still be used to give the levered value of the firm by adjusting the unlevered value. This procedure is called adjusted present value (APV).

The difficulty in applying the formula is that it requires an estimate of $R_A$. This is obtained by either unleveraging the WACC using one of the formulae given in Table 3, or estimating the asset beta. The general formula that can be used to unleverage betas is given in Appendix D. All these formulae implicitly make assumptions about the riskiness of the debt tax savings of the firms from which these estimates are obtained. In principle, therefore, the estimate of $R_A$ should be obtained from firms in the same industry as the company being valued, for which the assumptions underlying the formulae in Table 3 apply.
9 Discount rates for riskless cash flows.

One area which sometimes gives rise to confusion is the discounting of riskless flows. When valuing such cash flows, we are interested in either the discount rate that shareholders should apply to these flows if they were financed entirely with equity, or the appropriate tax-adjusted rate for the flows including their ability to generate tax savings from leverage. The first is the rate that should be applied in an APV calculation. The second is the equivalent of the WACC for riskless flows.

First, consider a riskless cash flow equal to $C_T$ that has already been taxed at $T_C$ and is paid out to the representative investor as an equity flow. Then the investor will receive $C_T(1 - T_{PE})$ and discount this net flow at $R_F(1 - T_{PD})$. $R_F(1 - T_{PD})$ is his after-tax riskless rate, so he values all net of tax riskless cash flows at this rate. The combined effect is that the cash flow $C_T$ is discounted at $R_F(1 - T_{PD})/(1 - T_{PE})$. The discount rate to use depends on the assumption about $T_{PD}$ and $T_{PE}$. For instance, if $T_{PD} = T_{PE} = 0$, riskless cash flows to equity are discounted at $R_F$. So the value of a tax saving equal to $T_C I$ in perpetuity is $T_C I / R_F$, which is $T_C D$.

This apparent complexity, where the discount rate appropriate to riskless cash flows appears to depend on the assumption about the representative investor, disappears if we use the tax-adjusted discount rate. This is the discount rate that incorporates the tax effect of borrowing, as does the WACC. In general for a risky project the tax-adjusted rate (the equivalent of the ‘WACC’) depends on the amount of incremental debt capacity of the project and the assumption about the net tax saving to debt, $T_S$. But in the case of a riskless cash flow, the tax and leverage adjusted discount rate does not depend on $T_S$ as long as the incremental borrowing capacity it adds to the firm is 100% of the cashflow’s value. In that case, the tax-adjusted discount rate is $R_F(1 - T_C)$ regardless of the assumption about $T_S$. This is the result referred to in Brealey and Myers (2003) and first shown by Ruback (1986). Regardless of the value of $T_S$, riskless cash flows can be valued, including the tax-impact of the debt they support, simply by discounting their after-corporate-tax level by $R_F(1 - T_C)$. 

10 Alternative Assumptions

In some situations it is appropriate to use a different set of assumptions to either the standard MM or ME assumptions. Three particular cases are where debt capacity is constrained (for instance by covenants), highly leveraged transactions (HLT’s), and non-tax-paying situations.

10.1 Constrained debt

In the case where debt capacity is constrained and the firm has already borrowed to the limit, then the 100% debt capacity of riskless flows no longer applies and the increased debt capacity resulting from an extra investment is zero. So all cash flows should be evaluated at the all-equity required rate of return appropriate to the risk level.

10.2 Highly Leveraged Transactions

In highly leveraged transactions it is unreasonable to believe that the interest charges will always save taxes. So the assumption that the tax saving is equal to the tax rate multiplied by the interest charge may no longer be true.

An alternative, used by Kaplan and Ruback, (1995) is to assume that the tax shield has the same risk as the firm’s assets. In this case, the tax shield is discounted at the firm’s all-equity cost of capital $R_A$. In this case:

$$V_L = V_A + \frac{IT_S}{R_A}. \quad (39)$$

If one assumes that $T_{PE} = T_{PD} = 0$ then:

$$V_L = \frac{C(1 - T_C) + IT_S}{R_A}. \quad (40)$$

This is the procedure of Kaplan and Ruback where they define the numerator of (19) as the ‘enterprise cash flow’ and then use (19) as ‘compressed APV’. These formulae are for perpetual debt. They can be written in a more general fashion, allowing for interest payments to vary over time. One of the main applications of this approach is leveraged buyouts, where debt levels tend to be declining over time.
In this ‘compressed APV’ procedure, the tax saving is discounted at the discount rate appropriate to
the firm’s assets, as in ME. But the Kaplan-Ruback procedure is not necessarily the same as ME as they
do not assume the same debt policy as ME. The reason for discounting the tax saving at $R_A$ in ME is that
debt is always proportional to the value of the firm’s assets. In Kaplan and Ruback this is not used as the
motivation, and they use the ‘compressed APV’ procedure for any highly leveraged transaction regardless of
whether the ME debt policy is followed.

10.3 The possibility of no tax deductions

In some cases the tax position will be more complex than assumed in a single tax rate. For example, a firm
may face the possibility of not generating taxable income. In these cases the tax deductibility of interest
generates a cash flow tax saving only when taxable income is positive. So valuing the tax deduction involves
forecasting the expected future tax position of the company. In general, this valuation should be done using
option technology, as the payoff to the tax deduction will have non-linearities like those of options.

11 Practical estimation and use of the cost of capital

In practice, estimates of discount rates for use in valuation start from observation of inputs to the WACC.
These are:

Inputs to the cost of equity: $R_F, \beta_E, P$

Inputs to the WACC formula: $R_D, D, E, T_C$

Assumption about the effect of the tax saving: $T^*$

Many of these are observed with error, particularly $\beta_E, P, D, and T^*$. The errors in these inputs to the
discount rate are significant, and all discount rates for company valuation are consequently highly uncertain.
However, it is still worth being consistent in the treatment of tax in the discount rate, as this is one potential
source of error that can be avoided.

From these inputs, it is standard to calculate the cost of equity and the WACC. The formula that should
be used for the cost of equity is:
\[ R_E = \frac{R_F(1 - T_C)}{(1 - T^*)} + \beta_E P \]  

where:

\[ P = R_M - R_F \frac{(1 - T_{PD})}{(1 - T_{PE})} = R_M - R_F \frac{(1 - T_C)}{(1 - T^*)} \]

Note that these expressions involve \( T^* \), unless \( T^* = T_C \). If one makes the judgement that \( T^* \) is not equal to \( T_C \), then the standard pre-tax version of the CAPM does not apply, and these expressions, with an adjusted riskless rate, should be used instead. As we have seen above, it is unlikely that \( T^* = T_C \) under an imputation system, so the riskless rate should always be adjusted in this way in an imputation tax system.

This also raises issues of how the market risk premium should be estimated, as the correct premium to use is one that is estimated relative to an adjusted riskless rate. For the US, in the period 1926-87, the historical average market risk premium was 7.7\% when measured relative to the gross treasury bill rate and 9.4\% relative to the t-bill rate net of \( T_C \). The former is the appropriate historical average if one uses the MM assumptions, and the latter if one uses the Miller assumptions.

Once the cost of equity is calculated, one can either use the WACC formula or calculate the asset beta from one of the formulae in Table 3. Note that the standard asset beta formula:

\[ \beta_A = \beta_D (D/V_L) + \beta_E (E/V_L) \]  

is the special case of the ME asset beta formula when \( T^* = T_C \).\(^{12}\)

To do this one needs the debt beta. The cost of debt is related to its beta by:

\[ R_D = R_F + \beta_D P \]

When one uses a particular value for \( R_D \) in the WACC formula, it is consistent with the CAPM only if

\(^{12}\)This is the relationship between pre-tax betas. If we use post-tax betas, then (43) will always be correct with continuous readjustment of leverage, since the tax rates of the representative investor will be contained in the post-tax debt beta.
the $\beta_D$ used satisfies this formula. For consistency, therefore:\(^{13}\)

$$\beta_D = (R_D - R_F)/P \quad (45)$$

If this debt beta is not used, then asset betas will not be consistent with the WACC.

Given these inputs, there are several routes by which one can include the tax effect of leverage in a valuation:

1. Calculate the WACC and use it to discount cash flows of the same risk as the firm.
2. Calculate the WACC, unleverage it and then releverage it to a new debt level.
3. Calculate the WACC, unleverage it and use this in an APV calculation.
4. Calculate $\beta_A$, and then releverage it to a new debt level either by releveraging $R_A$, or releveraging $\beta_E$ and calculating a new WACC.
5. Calculate $\beta_A$, then calculate $R_A$. Use this in an APV calculation.

Leveraging and unleveraging these rates almost always involves use of either the MM or the ME formulae. These are shown in Table 3 for the general case where $T^*$ is not equal to $T_C$, and in Table 4 for the case where $T^* = T_C$. Whichever choice one makes, it is important to be consistent. The same assumption should be used for unleveraging and releveraging.

The assumption made should be the one that reflects the leverage policy that the company is actually following. In most cases, this is likely to be closer to ME than to MM. Illustrations of the errors that can arise from inappropriate calculations is shown by the following examples. The base situation is given in Table 1. This describes a fairly typical company with 30% debt, a debt spread of 1%, and an equity beta of one. The corporate tax is 30% and $T^*$ is 20%, so that most, but not all, of the corporate tax flows through as a tax saving from interest.

The first column of Table 2 shows the results of calculations using ME for this firm. It calculates the required return on equity and the WACC. From the WACC it derives $R_A$. Alternatively, the asset beta is derived from the debt and equity beta and then used to calculate $R_A$. It does not matter which route is used,

\(^{13}\)In fact, the spread between the promised return on corporate debt and the riskless interest rate includes components related to tax, liquidity, and non-beta risk as well as beta risk. However, this formula ensures consistency in the rates used. For a more complete analysis of debt spreads, see Cooper and Davydenko (2004).
as they are consistent if the formulae in Table 3 are used. The unleveraged return, $R_A$, is then releveraged back to 30% debt, and the original WACC results, as it should. Finally, the rate is leveraged to a 60% debt ratio. This assumes, for illustrative purposes, that the debt spread remains constant at 1%.

Table 1: Assumptions for illustrative calculations

This table shows the assumptions used for the illustrative calculations in Table 2.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_F$</td>
<td>5%</td>
</tr>
<tr>
<td>$\beta_E$</td>
<td>1.0</td>
</tr>
<tr>
<td>$P$</td>
<td>5%</td>
</tr>
<tr>
<td>$R_D$</td>
<td>6%</td>
</tr>
<tr>
<td>$E$</td>
<td>0.7</td>
</tr>
<tr>
<td>$D$</td>
<td>0.3</td>
</tr>
<tr>
<td>$T_C$</td>
<td>0.3</td>
</tr>
<tr>
<td>$T^*$</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Table 2: Discount rates and betas resulting from applying the ME formulae

This table shows the outputs from various calculations related to the cost of capital.

The first column shows the correct values resulting from consistent application of the ME formula.

The other columns show the values resulting from various common errors.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Assuming $T^* = T_C$</th>
<th>Assuming $\beta_D = 0$</th>
<th>Using MM</th>
<th>Using ME $\beta_A$ and MM releveraging</th>
</tr>
</thead>
<tbody>
<tr>
<td>$WACC$</td>
<td>7.82%</td>
<td>8.26%</td>
<td></td>
<td>7.82%</td>
<td>7.82%</td>
</tr>
<tr>
<td>$R_E$</td>
<td>9.38%</td>
<td>10.00%</td>
<td></td>
<td>9.38%</td>
<td></td>
</tr>
<tr>
<td>$\beta_A$</td>
<td>0.75</td>
<td>0.76</td>
<td>0.70</td>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>$R_A$</td>
<td>8.14%</td>
<td>8.80%</td>
<td>7.88%</td>
<td>8.32%</td>
<td>8.14%</td>
</tr>
<tr>
<td>$R_L(30%)$</td>
<td>7.82%</td>
<td>8.26%</td>
<td>7.61%</td>
<td>7.82%</td>
<td>7.65%</td>
</tr>
<tr>
<td>$R_L(60%)$</td>
<td>7.51%</td>
<td></td>
<td></td>
<td>7.32%</td>
<td>7.16%</td>
</tr>
</tbody>
</table>
The other columns of Table 2 show the results of adopting inconsistent procedures. The first is the result of assuming that $T^*$ is equal to 30%, when it is 20%. This results in large errors in all rates, as the intercept of the CAPM is different for the two assumptions. One group of countries where this is important are those with imputation systems, where there is a natural presumption that $T^*$ is less than the full corporate tax rate. Another is countries where dividend income is taxed at a rate lower than that for normal income, which is the new US situation.

The illustration in Tables 1 and 2 assumes that the difference between $T^*$ and $T_C$ is 10%. For many countries, the imputation tax effect is larger than this and the effect on discount rates will also be larger. There are two ways around this problem. One is to estimate the value of $T^*$ and then estimate a value for the market risk premium that is consistent with this using (42). The other is to estimate the required return on equity using a variant of the dividend growth model. This involves many assumptions, but does at least avoid an assumption about the tax rate of the representative investor, as it estimates directly the after-corporate-tax required return.

The third column of numbers in Table 2 shows the effect of assuming that the debt beta is zero. This has an impact of 0.26% on $R_A$. This can be significant in some regulatory and valuation contexts. The effect would be larger for a more highly leveraged firm.

The fourth column shows the effect of using the MM formulae from Table 4 rather than the ME expressions. If the rate to be used has the same leverage as the WACC, it does not matter which approach is used. If, however, $R_A$ is used, then the error from unleveraging it using the MM expression is 0.18%. The ME rate is lower, because it assumes that more of the equity beta is generated by risk from the present value of future tax savings from interest. A similar magnitude of error, in the other direction results if the rate is releveraged to double the leverage of the firm using MM rather than ME.

The final column shows the effect of a commonly used procedure. This is to use the ME asset beta formula in conjunction with the MM releveraging formula. This results in an error of 0.17% for the discount rate at 30% leverage and 0.35% at 60% leverage.
12 Summary

This note has summarised the relationships between values, rates of return and betas that depend on taxes. It has extended the standard analysis to include the effect of risky debt. A consistent approach to this area involves understanding how basic assumptions feed through into the formulae that are used. Inconsistent application of these formulae can result in errors in estimated rates of return that are significant.

The note has also dealt extensively with the effects of an imputation system. Formulae for the tax saving from debt, and for required rates of return are different for classical and imputation systems.
This table shows the important relationships for the extended MM and the ME assumptions. All the rates apply to cash flows after corporate but before investor taxes. The version of ME used assumes instantaneous readjustment of the leverage ratio.

<table>
<thead>
<tr>
<th>Assumptions</th>
<th>EXTENDED MM</th>
<th>MILES-EZZELL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flows</td>
<td>Perpetuities</td>
<td>Any cash flow profile</td>
</tr>
<tr>
<td>Amount of debt</td>
<td>Constant debt</td>
<td>Constant proportional leverage</td>
</tr>
<tr>
<td>Value</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_L$</td>
<td>$V_A + T^* D$</td>
<td>$V_A + PV(Tax shield)</td>
</tr>
<tr>
<td>Rates</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_L$</td>
<td>$R_A(1 - T^*(D/V_L))$</td>
<td>$R_A - R_D[(1 - T_C)/(1 - T^<em>)]T^</em>(D/V_L)$</td>
</tr>
<tr>
<td>$R_E$</td>
<td>$R_A + <a href="D/E">R_A(1 - T^*) - R_D(1 - T_C)</a>$</td>
<td>$R_A + <a href="D/E">R_A - R_D(1 - T_C)/(1 - T^*)</a>$</td>
</tr>
<tr>
<td>$R_L$ for riskless flow</td>
<td>$R_F(1 - T_C)$</td>
<td>$R_F(1 - T_C)$</td>
</tr>
<tr>
<td>Betas</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_A$</td>
<td>$\beta_D(1 - T_C)(D/(V_L - T^* D)) + \beta_E(E/(V_L - T^* D))$</td>
<td>$\beta_D<a href="D/V_L">(1 - T_C)/(1 - T^*)</a> + \beta_E(E/V_L)$</td>
</tr>
<tr>
<td>$\beta_E$</td>
<td>$\beta_A + (\beta_A(1 - T^*) - \beta_D(1 - T_C))(D/E)$</td>
<td>$\beta_A + (\beta_A - \beta_D[(1 - T_C)/(1 - T^*)])(D/E)$</td>
</tr>
<tr>
<td>$\beta_A$ (zero beta debt)</td>
<td>$\beta_E[E/(V_L - T^* D)]$</td>
<td>$\beta_E(E/V_L)$</td>
</tr>
</tbody>
</table>

14 To compare these formulae with those in Taggart (1991), make the substitution $R_{FE} = \frac{R_E(1 - T_C)}{(1 - T^*)}$. We prefer to leave the dependence on $T^*$ in the formulae explicit, rather than embedded in the definition of $R_{FE}$, as in Taggart.
Table 4: Summary of useful relationships assuming no investor taxes

This table shows the important relationships for the standard MM and the ME assumptions. All the rates apply to cash flows after corporate but before investor taxes. The version of ME used assumes instantaneous readjustment of the leverage ratio.

<table>
<thead>
<tr>
<th></th>
<th>EXTENDED MM</th>
<th>MILES-EZZELL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assumptions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cash flows</td>
<td>Perpetuities</td>
<td>Any cash flow profile</td>
</tr>
<tr>
<td>Amount of debt</td>
<td>Constant debt</td>
<td>Constant proportional leverage</td>
</tr>
<tr>
<td>Value</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( V_L )</td>
<td>( V_A + T_C D )</td>
<td>( V_A + PV(Tax\ shield) )</td>
</tr>
<tr>
<td>Rates</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( WACC(R_L) )</td>
<td>( R_D(1 - T_C)(D/V_L) + R_E(E/V_L) )</td>
<td>( R_D(1 - T_C)(D/V_L) + R_E(E/V_L) )</td>
</tr>
<tr>
<td>( R_L )</td>
<td>( R_A(1 - T_C(D/V_L)) )</td>
<td>( R_A - R_D T_C(D/V_L) )</td>
</tr>
<tr>
<td>( R_E )</td>
<td>( R_A + [R_A - R_D](D(1 - T_C)/E) )</td>
<td>( R_A + <a href="D/E">R_A - R_D</a> )</td>
</tr>
<tr>
<td>( R_L ) for riskless flow</td>
<td>( R_F(1 - T_C) )</td>
<td>( R_F(1 - T_C) )</td>
</tr>
<tr>
<td>Betas</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \beta_A )</td>
<td>( \beta_D(1 - T_C)(D/(V_L - T_C D)) + \beta_E(E/(V_L - T_C D)) )</td>
<td>( \beta_D(D/V_L) + \beta_E(E/V_L) )</td>
</tr>
<tr>
<td>( \beta_E )</td>
<td>( \beta_A + (\beta_A - \beta_D)(D(1 - T_C)/E) )</td>
<td>( \beta_A + (\beta_A - \beta_D)(D/E) )</td>
</tr>
<tr>
<td>( \beta_A ) (zero beta debt)</td>
<td>( \beta_E(E/(V_L - T_C D)) )</td>
<td>( \beta_E(E/V_L) )</td>
</tr>
</tbody>
</table>
13 References


14 Appendix A: Notation

Cash flows and values:
- $C$: the pre-tax cash flow to the company
- $I$: total interest charges
- $D$: the market value of debt
- $E$: the market value of equity
- $V_L$: the total value of the leveraged firm
- $L = D/V_L$: the amount of leverage
- $V_A$: the value of the unleveraged firm
- $V_T$: the value of the tax saving
- $\alpha$: the payout ratio
- $c_D$: the after-tax investor flow from debt per dollar of corporate pre-tax cash flow
- $c_E$: the after-tax investor flow from equity per dollar of corporate pre-tax cash flow

Tax Rates:
- $T_C$: corporate tax rate
- $T_{PE}$: investor tax rate on equity
- $T_{PD}$: investor tax rate on debt
- $T_S$: net tax saving from $1$ of interest equal to: $T_S = (1 - T_{PD}) - (1 - T_C)(1 - T_{PE})$
- $T^*$: $T^* = T_S / (1 - T_{PD})$, the value increase from $1$ of debt under MM.
- $T_I$: imputation rate
- $T^I_C$: effective corporate tax rate with imputation
- $T^I_{PE}$: effective investor tax rate with imputation
- $T_{PED}$: the tax rate on gross dividends
- $T_{PEC}$: the tax rate on capital gains
Required returns:

\[ R_F \] riskfree rate
\[ R_E \] required return on equity after corporate tax
\[ R_A \] required return on unlevered equity after corporate tax
\[ R_D \] required return on firm debt
\[ WACC \] weighted average cost of capital
\[ R_{TS} \] discount rate for debt tax saving
\[ R'_E \] required return on equity after investor tax
\[ R'_A \] required return on unlevered equity after investor tax
\[ R'_D \] required return on debt after investor tax
\[ R'_E^I \] required return on equity before investor tax under imputation

CAPM inputs:

\[ \beta_E \] beta of pre-tax returns on equity
\[ \beta_D \] beta of pre-tax returns on debt
\[ \beta_A \] beta of pre-tax returns on unleveraged equity
\[ \beta_{TS} \] beta of tax saving from interest
\[ \beta'_E \] beta of after-tax returns on equity
\[ \beta'_D \] beta of after-tax returns on debt
\[ \beta'_A \] beta of after-tax returns on unlevered equity
\[ P' \] the market risk premium after-tax for the representative investor
\[ P \] the market risk premium before investor tax
15 Appendix B: Relationships between returns and betas for MM

Relationships between rates

WACC and $R_A$

\[
\text{WACC} = \frac{R_D(1 - T_C)D}{V_L} + \frac{R_EE}{V_L} = \frac{I(1 - T_C) + (C - I)(1 - T_C)}{V_L} \tag{46}
\]
\[
= \frac{C(1 - T_C)}{V_L} = \frac{R_AV_A}{V_L}
\]

Using $V_A = V_L - T^* D$:

\[
\text{WACC} = \frac{R_A(V_L - T^* D)}{V_L} = R_A[1 - T^* D/V_L] \tag{47}
\]

$R_E$ and $R_A$

\[
\text{WACC} = R_E(E/V_L) + R_D(1 - T_C)(D/V_L) \tag{48}
\]

Rearranging:

\[
R_E = (V_L/E) \text{ WACC} - R_D(1 - T_C)(D/E) \tag{49}
\]
\[
= (V_L/E)(R_A - R_A T^* D/V_L) - R_D(1 - T_C)(D/E)
\]
\[
= (E/E) R_A + (D/E) R_A (1 - T^*) - R_D(1 - T_C)(D/E)
\]
\[
= R_A + (D/E) [R_A(1 - T^*) - R_D(1 - T_C)]
\]

Relationships between betas and returns

The representative investor sets returns so that after-tax returns are in equilibrium. However, the CAPM is usually stated in terms of pre-tax betas and risk premia. This section uses the after-tax CAPM to derive the pre-tax version that is consistent with the assumptions about the tax saving on debt.
The Relationship between pre-tax and post-tax betas

Assuming that the market portfolio consists of only equities and not risky debt:

\[
\beta_E' = \frac{\text{Cov}(R_E(1 - T_{PE}), R_M(1 - T_{PE}))}{\text{Var}(R_M(1 - T_{PE}))} = \frac{\text{cov}(R_E, R_M)}{\text{var}(R_M)} = \beta_E
\]  

(50)

similarly:

\[
\beta_A' = \beta_A \text{ and } \beta_{TS}' = \beta_{TS}
\]  

(51)

and:

\[
\beta_D' = \frac{\text{Cov}(R_D(1 - T_{PD}), R_M(1 - T_{PE}))}{\text{Var}(R_M(1 - T_{PE}))} = \frac{(1 - T_{PD})}{(1 - T_{PE})} \beta_D
\]  

(52)

The relationship between after-tax expected returns and betas

\[
R_E(1 - T_{PE}) = R_F(1 - T_{PD}) + \beta_E P'
\]  

(53)

\[
R_A(1 - T_{PE}) = R_F(1 - T_{PD}) + \beta_A P'
\]  

\[
R_D(1 - T_{PD}) = R_F(1 - T_{PD}) + \beta_D \frac{(1 - T_{PD})}{(1 - T_{PE})} P'
\]

so

\[
R_E = R_F \frac{(1 - T_{PD})}{(1 - T_{PE})} + \frac{\beta_E}{(1 - T_{PE})} P'
\]  

(54)

\[
R_A = R_F \frac{(1 - T_{PD})}{(1 - T_{PE})} + \frac{\beta_A}{(1 - T_{PE})} P'
\]

\[
R_D = R_F + \frac{\beta_D}{(1 - T_{PE})} P'
\]

The relationship between pre-tax expected returns and betas

We define the pre-tax equivalent of the post-tax market risk premium:

\[
P = P'/(1 - T_{PE}) = R_M - R_F \frac{(1 - T_{PD})}{(1 - T_{PE})} = R_M - R_F \frac{(1 - T_C)}{(1 - T^*)}
\]  

(55)
Note that this is not equal to the pre-tax premium measured relative to the gross interest rate. The equilibrium is set by returns after investor taxes, and the differential treatment of equity and debt for the representative investor is reflected in the relationship between pre-investor-tax returns on equity and debt. Substituting $P$ for $P'$ gives:

$$R_E = \frac{R_F(1 - T_C)}{(1 - T^*)} + \beta_E P = \frac{R_F(1 - T_{PD})}{(1 - T_{PE})} + \beta_E P$$  \hspace{1cm} (56)$$

$$R_A = \frac{R_F(1 - T_C)}{(1 - T^*)} + \beta_A P = \frac{R_F(1 - T_{PD})}{(1 - T_{PE})} + \beta_A P$$  \hspace{1cm} (57)$$

$$R_D = R_F + \beta_D P$$  \hspace{1cm} (58)$$

**Asset beta, equity beta and debt beta:**

The relationship between rates of return is given by:

$$R_D(1 - T_C)(D/V_L) + R_E(E/V_L) = R_A(1 - T^* D/V_L)$$  \hspace{1cm} (59)$$

Substituting $R_D, R_E$ and $R_A$ in this gives:

$$\beta_D(1 - T_C) \frac{D}{V_L} + \beta_E \frac{E}{V_L} = \beta_A(1 - T^* D/V_L)$$  \hspace{1cm} (60)$$
16 Appendix C: Derivation of the Miles-Ezzell (ME) formulae

The ME formula applies to any profile of cash flows as long as the company maintains constant market value leverage. It gives a relationship between the leveraged discount rate, $R_L$, and the unleveraged rate, $R_A$. We derive the formula for a firm with expected cash flows $C_t$, $t = 1, .., T$. Between these dates, leverage remains fixed. After each cash flow, leverage is reset to be a constant proportion, $L$, of the value of the firm.

The two rates are defined implicitly by the discount rates that give the correct unleveraged and leveraged values when the operating cash flows are discounted:

$$V_{At} = \sum_{t=t+1}^{T} C_t (1 - T_C)/(1 + R_A)^i$$  \hspace{1cm} t = 1, ...T \hspace{1cm} (61)$$

$$V_{Lt} = \sum_{i=t+1}^{T} C_i (1 - T_C)/(1 + R_L)^i$$  \hspace{1cm} t = 1, ...T \hspace{1cm} (62)$$

The relationship between $R_L$ and $R_A$ is derived by induction, starting at time $T - 1$. At that time, the only cash flow remaining is $C_T$. The unleveraged value of this is:

$$V_{AT-1} = C_T (1 - T_C)/(1 + R_A)$$  \hspace{1cm} (63)$$

This is the value of the last cash flow, including the associated tax deduction of the purchase price, $V_{AT-1}$.

From the leveraged firm, the representative shareholder will receive a cash flow after personal taxes of $C_T(1 - T_C)(1 - T_{PE}) + I_TT_S$. The first part of this cash flow is identical to that from the unleveraged firm and so has value $V_{AT-1}$, if it is associated with a tax deduction equal to $V_{AT-1}$. The second flow has risk equal to debt, and should be discounted at the after-tax rate appropriate to the debt of the firm, $R_D(1 - T_{PD})$. Relative to an investment in the unleveraged firm, he also gets an extra tax deduction equal to $(V_{LT-1} - V_{AT-1})$. This is discounted at his after-tax riskless rate. Using $I_T = D_{T-1}R_D$ and $D_{T-1} = LV_{LT-1}$ the resulting value of the leveraged firm is:

$$V_{LT-1} = V_{AT-1} + \frac{D_{T-1}R_DT_S}{1 + R_D(1 - T_{PD})} + \frac{(V_{LT-1} - V_{AT-1})T_{PE}}{1 + R_P(1 - T_{PD})}$$  \hspace{1cm} (64)$$
The third term in this expression is due to capital gains taxes, which are assumed to be paid every year.\textsuperscript{15} The tax basis is higher in the leveraged case, and capital gains taxes are reduced.

Following Taggart (1991), we define the required return on riskless equity from (27) as:

$$R_{FE} = \frac{R_F(1 - T_C)}{(1 - T^*)} = \frac{R_F(1 - T_{PD})}{(1 - T_{PE})} \quad (65)$$

Note that, if $T_{PD}$ and $T_{PE}$ are equal, then $R_{FE} = R_F$. Using this and $T^* = T_S/(1 - T_{PD})$, we can rearrange $(64)$ as:

$$V_{LT-1} = V_{AT-1} + \frac{D_{T-1}T^*R_{FE}R_D(1 + R_F(1 - T_{PD}))}{(1 + R_{FE})R_F(1 + R_D(1 - T_{PD}))} \quad (66)$$

At time T-1, $R_L$ and $R_A$ are defined by:

$$1 + R_L = C_T(1 - T_C)/V_{LT-1} \quad (67)$$

$$1 + R_A = C_T(1 - T_C)/V_{AT-1} \quad (68)$$

Combining (66)-(68) and using $D_{T-1} = LV_{LT-1}$, we get:

$$R_L = R_A - \frac{LT^*R_{FE}(1 + R_A)R_D(1 + R_F(1 - T_{PD}))}{(1 + R_{FE})R_F(1 + R_D(1 - T_{PD}))} \quad (69)$$

A similar argument shows that the same relationship holds at all dates prior to T-1.

If the period between rebalancing the leverage becomes short, this expression converges to:

$$R_L = R_A - LT^*R_{FE}(1 - T_C)/(1 - T^*) = R_A - LR_D\frac{T_S}{(1 - T_{PE})} \quad (70)$$

This is the expression shown in Table 3. Taggart (1991) implicitly assumes that corporate debt is riskless, and derives this expression with $R_F$ substituted for $R_D$.

\section*{17 Appendix D: Relationships between betas}

We can understand the relationships between betas intuitively in the following way. The leveraged firm’s assets are the same as those for the all-equity firm. The only differences are that the leveraged firm generates\textsuperscript{16} There is an emerging literature that introduces realistic treatment of capital gains taxes into the capital structure literature (see Lewellen and Lewellen (2004)). The implications of their results for practical valuation are not yet clear.
extra value through the tax saving from interest, and changes the after-tax risk of the cash flow stream by channeling some of it to debtholders rather than equityholders, which changes the associated tax treatment. The weighted average of the equity beta and the tax-adjusted debt beta for the leveraged firm must equal the asset beta adjusted for the effect of the tax saving:

\[ E\beta_E + D\frac{(1-T_{PD})}{(1-T_{PE})}\beta_D = \beta_A(V_L - V_{TS}) + V_{TS}\beta_{TS} \]  

(71)

where \( V_{TS} \) is the value of the tax shield and \( \beta_{TS} \) is its beta. The value \((V_L - V_{TS})\) is the all-equity value of the firm, which has beta equal to \( \beta_A \). The adjustment \( \frac{(1-T_{PD})}{(1-T_{PE})} \) to the debt beta reflects the fact, shown in (53), that the differential tax treatment of debt and equity results in a change in beta when cash flow is switched from equity to debt, even apart from the effect on the value of the firm.

With the extended MM assumptions, \( V_{TS} = T^*D, \beta_{TS} = \frac{(1-T_{PD})}{(1-T_{PE})}\beta_D \), and substitution using \( \frac{(1-T_{PD})}{(1-T_{PE})} = \frac{(1-T_C)}{(1-T^*)} \) yields:

\[ \beta_A = \beta_D(1-T_C)(D/(V_L - T^*D)) + \beta_E(E/(V_L - T^*D)) \]  

(72)

With the ME assumptions, \( \beta_{TS} = \beta_A \), giving:

\[ \beta_A = \beta_D[(1-T_C)/(1-T^*)](D/V_L) + \beta_E(E/V_L) \]  

(73)

These are the expressions shown in Table 3.