International Good Market Segmentation
and Financial Market Structure*

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Abstract

While financial markets have recently become more complete and international capital flows well liberalized, markets for goods remain segmented. To investigate how more complete security markets may relieve the effects of this segmentation, we examine a series of two-country economies with internationally segmented good markets, distinguished by the assumptions made on financial market completeness. Under homogeneity within both countries, the location and structure of financial markets is shown to be irrelevant in equilibrium, as long as markets are internationally complete. Under heterogeneity within one country, however, a domestic market in this country is needed for perfect risk-sharing within the country (an assumption typically made in the literature, so as to be able to substitute a representative agent for each country). The requirements on the completeness of this domestic market are shown to differ drastically depending on whether a single-period or a multiperiod model is employed, highlighting a specific role for financial innovation in a dynamic setting. This is because a multiperiod setup allows financial securities to be used for the spot trade of goods and, thus, alleviate some of the good market segmentation as international capital flows replace part of the trading in goods.

JEL classification numbers: F30, F36, G12, G15

Keywords: market segmentation, financial innovation, international capital flows, dynamic equilibrium.
1. Introduction

The past 30 years or so have been characterized by an unprecedented development in the menu of financial instruments available and an increased liberalization of capital movements.\(^1\) Progress in the free trade of goods, however, has been more limited, and there remain, even among developed countries, significant obstacles to the international shipment of goods.\(^2\) Even with perfect financial markets, such good market segmentation would impose a burden on risk sharing by restricting the consumption of gains from financial trades with foreigners. A natural question then arises as to the extent to which the improvement in financial transfers can alleviate the burden on risk sharing imposed by good market segmentation.

The international finance, good market segmentation literature (Dumas (1992), Uppal (1992, 1993), etc.) has not thus far focused on the role of financial markets; equilibrium is typically solved via a central planner problem, where each country is modeled as a single representative agent, and the security market need not be modeled explicitly. The justification for this is that the equilibrium is assumed to be “Pareto optimal” (in the sense of Dumas (1992)) due to the financial market being “complete”. There are, however, different notions of market completeness in an international setting. We shall refer to “international market completeness” if any cash-flow may be attained, but foreign securities may need to be used (and goods to be shipped). “Domestic market completeness” in a country is a more stringent requirement, defined as any cash-flow being attainable using the domestic securities alone.

Our objective is to examine the extent to which financial innovation may alleviate the effects of segmentation in the good market. A part of this is to identify the financial market structure needed to justify the solution of equilibrium by means of a central planner. We consider a pure exchange economy with two countries, one having a homogeneous population proxied for by a single agent, but the other inhabited by two agents, heterogeneous in their preferences and endowments. There is a single consumption good, but the market therein is segmented, in that agents can only consume goods located in their own country, and must incur (proportional) shipping costs to transfer goods across borders. Securities are defined not only by their pay-offs, but also by their location, i.e., whether they can be redeemed for goods located in country 1 or country 2. It will appear that, because of the good market segmentation, the location of

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\(^1\) A limited amount of financial innovation took place as early as the 1960s, but the pace of innovation quickened considerably from the 1970s on; for a survey, see Allen and Gale (1994). There was a simultaneous evolution toward financial integration in developed countries, with such moves as the removal of capital controls following the end of the Bretton Woods system and the development of Euro-currency markets (see, e.g., Frankel (1986), Halliday (1989)). The liberalization is more recent in emerging economies (from the second half of the 1980s on) but is now well-documented, as are its effects in terms of financial integration (see, e.g., the chronologies in Bekaert and Harvey (2000) and Henry (2000), and the estimates of effects on the cost of capital in Bekaert and Harvey (2000), Errunza and Miller (1998)).

\(^2\) Rogoff (1996) includes an overview of frictions in international trade; he estimates actual shipping costs to be approximately 10% for 1994. Tariffs have been declining but remain significant, ranging (on average) from 3 to 10% across countries (Whalley and Hamilton (1996, Table 3.1)). Rogoff (1996) concludes his analysis of the “purchasing power parity puzzle” by stating that there is no other satisfactory explanation for it than international good markets remaining quite segmented, due to the above frictions as well as nontariff barriers.
securities matters. Employing a general equilibrium approach, we solve for equilibrium under several financial market structures, distinguished by the securities that are available for trading; in all economies, however, markets are internationally complete. This assumption is consistent with our earlier observation that, after decades of innovation, investors now face a rich menu of financial instruments. Our approach is novel in that we explicitly model the security market, rather than deduce allocations from a central planner problem.

Our main concepts are first introduced in a single-period framework, with two states of nature and logarithmic utility for all agents. Comparison of the equilibria arising under various financial market structures reveals that, under intranational homogeneity, the location of securities is irrelevant as long as markets are internationally complete; agents use security prices so as to effectively share the shipping costs. Financial structure, however, matters under heterogeneity within one country: if that country’s domestic market is incomplete, its agents incur shipping costs when trading with each other, hence there is incomplete risk-sharing within the country and a central planner cannot be substituted. Thus, we provide a model in which financial innovation has real effects, even under international market completeness, establishing that securities can indeed relieve imperfections in markets for goods, a fact little studied thus far.

This single-period setup, however, leaves aside an important role for securities, because no security trade occurs after endowments of the good are received. In a multiperiod model, where goods and securities are traded concurrently, agents have the possibility to “ship financially”: for example, rather than ship goods from country 1 to country 2 (and incur shipping costs), an agent can purchase securities in country 1, exchange these for country-2 settled securities, and sell the latter for country-2 goods, as is done in real-life foreign exchange transactions. Unlike with physical shipping, goods “shipped” financially are traded at the same real exchange rate no matter the direction of shipping (importing or exporting); thus, financial shipping allows an effective linearization in the transfer of goods from one country to the other, but positive value securities in both countries are needed for this. Because of this, any multiperiod model exhibits radically different features from our single-period model. In this respect, our analysis provides an example where financial innovation plays a role specific to the dynamic setting (a feature missing in the existing literature, as pointed out by Allen and Gale (1994)). This striking difference between the single- and multi-period cases constitutes a highly unusual feature of our model.

Thus, for the bulk of the paper, we adopt a continuous-time setup. The segmentation makes the agents' optimization nonstandard. In the absence of any domestic financial market, an agent incurs shipping costs whenever he trades goods for securities. His dynamic budget constraint is therefore nonlinear in his consumption. He effectively faces a nonlinear price for consumption, and thus has a positive measure region (over his endowment) on which he does not trade. In the presence of a domestic market (whether complete or not), however, an agent has “financial shipping” as an alternative to the actual shipment of goods; this linearizes his problem as he now faces an identical price for consumption, whether he buys or sells goods.

Explicit characterization is provided for pertinent economic quantities: equilibrium interest
rates, market prices of risk, imports/exports and exchange rates. In the case of intranational homogeneity, the nature of the equilibrium is similar to the single-period case: the structure and location of financial markets is similarly irrelevant; as long as markets are internationally complete, a Pareto optimal allocation always obtains. Under intranational heterogeneity, however, thanks to the availability of financial shipping, the requirements for Pareto optimality are much less stringent than in the single-period case. We show that it is enough that financial shipping (or foreign exchange) be available to all agents within the heterogeneous country (with no restriction on financial flows); this ensures that an efficient allocation obtains within that country. Even a severely incomplete market is sufficient, as long as a positive value domestic security is available. In this respect, financial markets provide a remarkably effective relief of the burden imposed on risk-sharing by segmentation in the good market. Somewhat surprisingly, however, financial shipping only improves risk-sharing between agents from the same country.

Our work clearly emphasizes the importance of investigating the properties of equilibrium with three agents (a case largely ignored in the literature) as, unless aggregation to the two-agent case is possible, the equilibrium is radically different. Our work also provides support for the body of literature originating in Dumas (1992), and solving for equilibrium via a central planner problem. We establish this approach to be valid as long as (unconstrained) financial markets are internationally complete and at least one positive value security is available in both countries, a set of assumptions consistent with the recent evolution of the international financial system. Dumas (1992), in a setup somewhat similar to ours, considers an economy with homogeneous linear production technologies in two countries, allowing an incorporation of an intertemporal capital investment decision. Unlike in our pure-exchange model, he must employ dynamic programming in his solution and hence his results are largely numerical. His focus is also different from ours, in that he primarily examines the behavior of the real exchange rate and considers a single financial market structure whose importance in enabling an efficient allocation is largely ignored.

The modelling strategy of Dumas (1992) is applied by various authors to several issues in international finance. Uppal (1992, 1993) examines agents’ portfolio choice and international financial flows. His focus is on portfolio holdings, but he still solves for equilibrium by means of a central planner problem and does not examine the role of securities in making this possible. Sercu, Uppal and Van Hulle (1995), Hollifield and Uppal (1997) and Pavlova (2000) study other issues. Dumas and Uppal (2001), in a largely similar setting, but with recursive preferences, tackle a problem that can be seen as the mirror image of ours: rather than varying the financial market structure as we do, they vary the level of good market segmentation, and assess the benefits of international financial market integration in each case.

The literature on gains from international risk-sharing (e.g., Cole and Obstfeld (1991), Tesar (1995)) is also related to our work, in that it compares equilibria arising under different financial market structures, albeit in the polar cases of no international financial market and perfect integration. These papers also typically assume that a representative agent can be substituted for each country. As our work shows, however, this may not be justified in the absence of
international financial trade. Indeed, our work suggests that foreign financial markets may be used by agents from a single country to trade risk with each other. If correct, this means that this body of literature may be understating the benefits of financial integration. This could explain why the gains from integration are typically estimated to be so small (about 0.20 percent of output according to Cole and Obstfeld (1991)). This underscores the potential importance of allowing for intranational heterogeneity in international finance, a novelty of this work.

Our results could also help explain why, in the recent past, the growth in international financial flows has far outpaced the growth in international good trade, a fact often viewed as paradoxical. In our model, as financial markets become more complete and international financial flows are liberalized, some good trading is replaced by financial trading, as agents from the same country do not need to ship goods to trade risk with each other any more.

The rest of the paper is organized as follows. Section 2 provides the results of the single-period model. Section 3 presents the continuous-time setup, while in Section 4 our agents’ optimization problems are solved. Sections 5 and 6 analyze the equilibrium under, respectively, intranational homogeneity and intranational heterogeneity. Section 7 concludes, Appendix A provides details on the single-period case, and Appendix B all proofs for the continuous-time case.

2. A One-Period Example

This section introduces some of the main ideas of the paper in the simplest possible setting, a single-period economy with two states of nature and logarithmic utility for all agents. Only the setup and an example are presented here; more general results are relegated to Appendix A.

2.1. The Setup

We consider a single good, single period model under uncertainty, with two equally probable states of nature, \(a, b\). There are two countries 1, 2, populated by, respectively, one \((i = 1)\) and two \((i = 2, 3)\) agents (the single agent in country 1 being understood as a proxy for a homogeneous population). There are two dates, \(t = 0\) (when agents choose their portfolio holdings) and \(t = 1\) (when agents consume). Each agent maximizes the expected logarithm of time-1 consumption. Agents receive random time-1 endowments of the good located in their own country, \(\epsilon_{ai}, \epsilon_{bi}\):

![Diagram of Country 1 and Country 2 with endowments \(\epsilon_{ai}, \epsilon_{bi}\) and shipping costs.

Agents can only consume good located in their home country. There are shipping costs for transportation of the good between the two countries: if one unit of good is being shipped from a

\(^3\)Obstfeld (1994) obtains far higher estimates, but these are mostly due to the indirect effects, in a production economy, of international risk-sharing, via the switch it causes to riskier, more profitable projects.
country, only \( k \) unit arrives (immediately) in the other, where \( 0 < k < 1 \). Hence, the good market is segmented.

We define four zero-net supply securities, \( a, b, a^*, b^* \), with (relative) prices \( p_a, p_b, p_{a^*}, p_{b^*} \). Security \( a \) pays one unit of good located in country 1 if state \( a \) occurs and zero otherwise, security \( b \) pays one unit of good located in country 1 if state \( b \) occurs and zero otherwise, while \( a^* \) and \( b^* \) pay similarly, in units of good located in country 2:

We consider Economies I, I*, II, III, differing in the securities that are available for trading.

<table>
<thead>
<tr>
<th>Economy</th>
<th>Available Securities</th>
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<tbody>
<tr>
<td>Economy I</td>
<td>( a, b )</td>
</tr>
<tr>
<td>Economy I*</td>
<td>( a^<em>, b^</em> )</td>
</tr>
<tr>
<td>Economy II</td>
<td>( a, b, a^* )</td>
</tr>
<tr>
<td>Economy III</td>
<td>( a, b, a^<em>, b^</em> )</td>
</tr>
</tbody>
</table>

Economies I and I* have a complete financial market in one country and no financial market in the other. Economy II has a complete financial market in one country (1) and an incomplete market in the other, while Economy III has complete domestic markets in both countries. In all economies, however, markets are internationally complete for all agents, in the sense that any cash-flow can be perfectly replicated (at a cost that may differ across countries). The progression Economy I \( \rightarrow \) Economy II \( \rightarrow \) Economy III allows us to evaluate the effect of introducing more financial markets specifically in a country with heterogeneous agents. In particular, we may evaluate the extent to which financial securities allow circumvention of the good market segmentation (shipping costs). Economy I* is introduced for comparison with Economy I in Section 2.2.

2.2. Equilibrium under Intranational Homogeneity

We now assume homogeneity in the distribution of endowments across states within country 2 (i.e., \( \epsilon_{a2}/\epsilon_{b2} = \epsilon_{a3}/\epsilon_{b3} \)), hence effectively populating it with a single representative agent \( (i = 2) \). Appendix A provides the equilibrium quantities, which reveal that all Economies I-III yield the same allocations. Hence, under intranational homogeneity, the financial market structure has no impact on the real equilibrium quantities. This is somewhat counterintuitive: one might expect an agent with a complete domestic financial market to be better off than an agent with no domestic market, because the former does not incur shipping costs to trade. This is not the case, however, because security prices adjust to share these costs among agents in an efficient way. For example, in the case where country 1 imports, and country 2 exports, on moving from
Economy I (financial markets in country 1 only) to Economy I* (financial markets in country 2 only), the relative price of the securities paying off in state $b$ versus in state $a$ changes from

$$
\xi^I \equiv \frac{p_b}{p_a} = \frac{k\epsilon_{a1} + k^2\epsilon_{a2}}{k\epsilon_{b1} + k\epsilon_{b2}}
$$

to

$$
\xi^{I*} \equiv \frac{p_{b*}}{p_{a*}} = \frac{\epsilon_{a1} + k\epsilon_{a2}}{k^2\epsilon_{b1} + k\epsilon_{b2}} = \frac{\xi^I}{k^2}.
$$

This increase in the relative price of consumption in state $b$ benefits the country that is selling consumption in state $b$, i.e., country 1, which is also the country one would expect to be hurt by the change in the location of financial markets. So the change in relative prices among the available securities compensates for the incurred shipping costs. The allocation can easily be checked to be “Pareto-optimal” (in the sense of Dumas (1992)), in that it solves a central planner’s problem with constant weights for the two countries.

### 2.3. Equilibrium under Intranational Heterogeneity

We now return to the general case of heterogeneity within country 2. The neutrality of financial market structure under intranational homogeneity motivates this generalization. A numerical example only is developed here, and general results are relegated to Appendix 1. To make our point clear, we choose endowments that are highly heterogeneous among the country-2 agents:

<table>
<thead>
<tr>
<th>Country 1</th>
<th>Country 2</th>
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<tbody>
<tr>
<td>$i = 1$</td>
<td>$i = 2$</td>
</tr>
<tr>
<td>$\epsilon_{a1} = 5$</td>
<td>$\epsilon_{a2} = 5$</td>
</tr>
<tr>
<td>$\epsilon_{b1} = 2$</td>
<td>$\epsilon_{b2} = 1$</td>
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We further assume that $k = 0.9$, and adopt the financial market structure of Economy I (a complete market in country 1, but no securities in country 2.)

Equilibrium consumption allocations, as detailed in Appendix A, are:

<table>
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<tr>
<td>$i = 1$</td>
<td>$i = 2$</td>
</tr>
<tr>
<td>$c_{a1} = 4.2$</td>
<td>$c_{a2} = 3.6$</td>
</tr>
<tr>
<td>$c_{b1} = 2.4$</td>
<td>$c_{b2} = 1.7$</td>
</tr>
</tbody>
</table>

Country-2 agents’ consumption in the two states reveals their marginal rates of substitution ($u'(c_a)/u'(c_b)$) to be different (as $3.6/1.7 \neq 3.8/2.7$), and so the allocation within country 2 is inefficient, in spite of the completeness of financial markets (at the international level). Thus, it does not prove possible to substitute a representative agent for the heterogeneous country, and the results of the previous subsection cannot be generalized: the financial structure does matter,
even under internationally complete markets. A fortiori, the allocation across countries does not solve a central planner’s problem either.

These inefficiencies arise from the fact that, under incompleteness of the domestic financial market, agents 2 and 3 may indulge in different directions of shipping and thus face effectively different prices for consumption. This is the case in the above example: agent 3, who has a low endowment in state \( a \) and hence is a net purchaser of consumption in this state, needs to purchase \( 1/k \) units of \( a \) to increase his consumption by one unit, hence facing a state price equal to \( p_a/k \); agent 2 (short in \( a \)) faces a state price equal to \( kp_a \). The inefficiency that occurs manifests itself in two ways: the inequality in marginal rates of substitution across agents 2 and 3; and the “wasteful shipping”. Goods are shipped back and forth wastefully when one agent in country 2 is importing and the other is simultaneously exporting: due to the absence of a complete domestic financial market, agents 2 and 3 need to go abroad to trade with each other.

Comparison across economies (as detailed in Appendix A) reveals financial innovation to relieve these inefficiencies. Less and less wasteful shipping occurs as new securities are introduced: in Economy II, wasteful shipping occurs only in state \( b \) since a country 2 settled security paying off in state \( a \) is now available; in Economy III, no wasteful shipping occurs, as agents 2 and 3 do not need to go abroad to trade with each other any more. Then, an efficient allocation obtains within country 2, and the allocation across countries solves a central planner problem, as in the previous case of homogeneous countries. A complete market within any heterogeneous country is needed for this result. An incomplete market (as in Economy II) provides only partial circumvention of the inefficiencies.

In short, to obtain an efficient allocation across countries and to justify the use of a central planner, strong assumptions (either homogeneity or a complete domestic financial market within each country) are needed. The next subsection suggests, however, that this result is largely an artifact of the single-period setup; this will be verified in our multiperiod model.

2.4. “Financial Shipping” and the Dual Role of Securities in a Multiperiod Model

In this single period setup, agents decide on their portfolio at \( t = 0 \) and ship goods only at \( t = 1 \). The situation is different in a multiperiod model where agents may trade in securities at the same dates they ship goods. Then, in economies where securities are traded in both countries, each agent can, rather than “physically” shipping goods, do “financial shipping”: for example, rather than export goods from country 1 to country 2, an agent can exchange country-1 goods for country-1 financial securities, then exchange the country-1 securities for country-2 securities, then country-2 securities for country-2 good. Even though such spot good trades cannot be modeled independently of the financial market structure, they put only mild requirements on it.

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4Similar phenomena occur for an extended region of endowments; conditions for its occurrence are presented in Appendix A.
In particular, the availability of financial shipping is independent of market completeness: all that is needed is the presence in both countries of at least one security with a strictly positive value. Currencies typically satisfy this condition and are probably the most straightforward example of a security commonly employed for financial shipping (in foreign exchange transactions).

Hence, in a multiperiod model, securities play a dual role: as a tool for risk-sharing and as a substitute for the shipment of goods. Ignoring this, as a single-period model necessarily does, leads to inefficiencies that do not reflect any real-life imperfection. This suffices to make any multiperiod model qualitatively very different from the one-period model, and motivates our study of the continuous-time model to which we now turn.

3. The Continuous-Time Formulation

We consider a continuous-time, pure-exchange economy with a finite horizon $[0,T]$ and a single consumption good. The uncertainty is represented by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ (with $\mathcal{F} = \mathcal{F}_T$) on which is defined a one-dimensional Brownian motion $W$. The economy is populated by three agents, $i = 1$ (living in country 1) and $i = 2, 3$ (living in country 2), homogeneous in their (complete) information (represented by $\{\mathcal{F}_t\}$, the augmented filtration generated by $W$) and beliefs (represented by $\mathcal{P}$). All the stochastic processes introduced henceforth are assumed to be $\{\mathcal{F}_t\}$-progressively measurable, all equalities involving random variables hold $\mathcal{P}$-a.s., and all stochastic differential equations are assumed to have a solution.

3.1. Investment Opportunities

Investment opportunities consist of the following four securities. There are two zero-net supply “bonds” (money market accounts) with prices $M$ and $M^*$, settled in country 1 and country 2 respectively (meaning that they can be exchanged for goods located in countries 1 and 2 respectively). In addition, there are two risky securities, each also in zero net supply, representing claims to the exogenously specified dividend processes $\delta$ and $\delta^*$, and with prices $P$ and $P^*$, settled in countries 1 and 2 respectively. The numeraire for securities $M$, $P$ is the good located in country 1, and that for $M^*$, $P^*$ is the good located in country 2. Security prices have dynamics

\[
\begin{align*}
    dM(t) &= M(t)\mu(t)dt, \\
    dM^*(t) &= M^*(t)\mu^*(t)dt, \\
    dP(t) + \delta(t)dt &= P(t)\left[\mu(t)dt + \sigma(t)dW(t)\right], \\
    dP^*(t) + \delta^*(t)dt &= P^*(t)\left[\mu^*(t)dt + \sigma^*(t)dW(t)\right].
\end{align*}
\]

All price parameters ($\mu, \mu^*, \sigma, \sigma^*$) are to be determined endogenously in equilibrium.

As in the single period case, we consider four economies, I, I*, II, III:
<table>
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</tr>
<tr>
<td>Economy II</td>
<td>$M, P, M^*$</td>
</tr>
<tr>
<td>Economy III</td>
<td>$M, P, M^<em>, P^</em>$</td>
</tr>
</tbody>
</table>

In Economies I, II, III, country 1’s domestic financial market is complete; then, all cash-flows paid in country 1 can be valued using the country-1 state price density process $\xi$ with dynamics

$$d\xi(t) = -\xi(t) [r(t)dt + \theta(t)dW(t)],$$

where $\theta(t) \equiv (\mu(t) - r(t))/\sigma(t)$ denotes the market price of risk in country 1. This state price density is country-1 specific, in that it takes cash-flows paid in units of the good located in country 1 and yields a unique no-arbitrage price denominated in the same unit; accordingly, under standard regularity,

$$P(t) = \frac{1}{\xi(t)} E \left[ \int_t^T \xi(s)\delta(s)ds \bigg| \mathcal{F}_t \right];$$

and the deflated security gain processes $\xi P + \int_0^t \xi\delta dt$, $\xi M$ are $\mathcal{P}$-martingales. In Economies I* and III, we can similarly define a country-2 state price density process $\xi^*$ by

$$d\xi^*(t) = -\xi^*(t) [r^*(t)dt + \theta^*(t)dW(t)],$$

where $\theta^*(t) \equiv (\mu^*(t) - r^*(t))/\sigma^*(t)$ denotes the market price of risk in country 2. Since the determination of $\xi$, $\xi^*$ is sufficient to pin down all security prices, we focus our attention on these rather than the actual price processes $M, M^*, P, P^*$.

### 3.2. The Shipment of Goods and the Real Exchange Rate

While security markets are perfect, there are shipping costs for transportation of the good between the countries: if one unit of good is being shipped from a country, only $k$ unit arrives (instantly) in the other, where $0 < k < 1$. This (together with the fact that agents can only consume good located in their own country) makes the good market segmented. We denote the amounts shipped, assuming the viewpoint of country 1, by $x^E$ and $x^I$. $x^E(t) \geq 0$ is the amount of good exported, leaving country 1 at time $t$, and $x^I(t) \geq 0$ denotes the amount of good imported arriving in country 1.

Whenever securities are traded in both countries (Economies II and III), agents can freely exchange securities settled in one country for securities settled in the other (reflecting our observation that capital flows are now quite well liberalized). At the individual level, this is a substitute

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5 The shipping cost could be made specific to the direction of shipping (to account, for example, for the different nature of each country’s exports) at only the cost of additional notational complexity.
for the actual, “physical” shipment of goods, a notion we will refer to as “financial shipping”. The real exchange rate implicit therein (unique from no-arbitrage) is denoted $p(t)$ (expressed in units of good located in country 1 per unit of good located in country 2): for example, agents can exchange one unit of bond $M^*$ for $(M^*(t)/M(t)) p(t)$ unit(s) of bond $M$ or $(M^*(t)/P(t)) p(t)$ unit(s) of stock $P$. The exchange rate implicit in physical shipping equals either $k$ or $1/k$, depending on the direction of shipment; whereas the rate for financial shipping equals $p$ no matter the direction of shipment. No-arbitrage (that would be exploited by simultaneously shipping goods “financially” in one direction and “physically” in the other) implies:

$$k \leq p(t) \leq \frac{1}{k}. \tag{3.2}$$

The real exchange rate $p$ will be shown in equilibrium to have dynamics:

$$dp(t) = p(t) [\mu_p(t)dt + \sigma_p(t)dW(t)].$$

### 3.3. Agents’ Endowments and Preferences

Each of the three agents in the economy, $i = 1$ (living in country 1) and $i = 2, 3$ (country 2), is endowed with the (bounded) stochastic endowment process $\epsilon_i$, with $\epsilon_i(t) > 0$, $\forall t$, expressed in units of the good located in his own country. $\epsilon_i$ has dynamics:

$$d\epsilon_i(t) = \mu_{\epsilon_i}(t)dt + \sigma_{\epsilon_i}(t)dW(t).$$

Agent $i$’s portfolio holdings, expressed, for each security, in amounts of the good of the same nationality, are denoted by $\pi_i \equiv (\pi_{Mi}, \pi_{M^*i}, \pi_{Pi}, \pi_{P^*i}, \pi_{Mi}, \pi_{P^*i})^\top$, with $\pi_{M^*i} \equiv \pi_{P^*i} \equiv 0$ in Economy I, with $\pi_{Mi} \equiv \pi_{Pi} \equiv 0$ in Economy $I^*$ and $\pi_{P^*i} \equiv 0$ in Economy II. In Economies I, II and III, we can express agent $i$’s wealth in units of the good located in country 1, a quantity that will be denoted $X_i \equiv \pi_{Mi} + \pi_{Pi} + p(\pi_{M^*i} + \pi_{P^*i})$; when securities are traded in country 2 (Economies $I^*$, II and III), $i$’s wealth can be measured in units of country-2 good, denoted $X_i^* \equiv \pi_{M^*i} + \pi_{P^*i} + (\pi_{Mi} + \pi_{Pi})/p$. When securities are traded in both countries and an exchange rate is thus well-defined, we have $X_i^*(t) \equiv X_i(t)/p(t)$. A consumption-portfolio pair $(\epsilon_i, \pi_i)$ is admissible if the associated wealth process $X_i$ is bounded below, satisfies $X_i(T) \geq 0$ (or $X_i^*(T) \geq 0$ in Economy $I^*$) and obeys a dynamic budget constraint (such as (4.1)) that depends on the economy under consideration. Agent $i$ derives time-additive, state-independent utility $u_i(\epsilon_i(t))$ from intertemporal consumption of the good located in his own country in $[0, T]$. The function $u_i(\cdot)$ is assumed to be three times continuously differentiable, strictly increasing, strictly concave, and to satisfy $\lim_{c \to 0} u_i'(c) = \infty$ and $\lim_{c \to \infty} u_i'(c) = 0$. We denote the inverse of the first derivative of $i$’s utility by $I_i(\cdot) \equiv (u_i')^{-1}(\cdot)$. Agent $i$’s optimization problem is to maximize $E \left[ \int_0^T u_i'(\epsilon_i(t)) dt \right]$ over all admissible $(\epsilon_i, \pi_i)$ for which the expected integral is well defined.
3.4. Equilibrium

**Definition 3.1.** An equilibrium is a price system \((\mu, r)\) in Economy I, \((\mu^*, r^*)\) in Economy I*, \((\mu, r, r^*, p)\) in Economy II, \((\mu, r, \mu^*, r^*, p)\) in Economy III and consumption-portfolio processes \((c_i, \pi_i)\) such that: (i) agents attain their optimal consumption-portfolio processes; (ii) the good markets in the two countries clear, i.e.,

\[
c_1(t) = \epsilon_1(t) + x^I(t) - x^E(t), \quad c_2(t) + c_3(t) = \epsilon_2(t) + c_3(t) + kx^E(t) - x^I(t)/k; \quad (3.3)
\]

(iii) security markets clear, i.e., \(\sum_i \pi_{ji} = 0\), for \(j \in \{M, P\}\) in Economy I, \(j \in \{M^*, P^*\}\) in Economy I*, \(j \in \{M, P, M^*\}\) in Economy II, \(j \in \{M, P, M^*, P^*\}\) in Economy III.

4. Agents’ Optimization

Agents’ optimization problems depend on their nationality and the economy (I-III) under consideration. An agent with no domestic financial market faces a nonlinear problem; his wealth dynamics are nonlinear in his consumption due to the shipping costs incurred whenever he trades in securities. However, in the presence of a domestic market, even incomplete, an agent faces a linear problem: “financial shipping” allows him to purchase or sell consumption at a single rate. For brevity and ease of notation, we only present Economies I, II, III. (Economy I* is symmetric.)

4.1. The Case of No Domestic Financial Market

When agent \(i\) has no domestic financial market (agents 2 and 3 in Economy I), he must ship goods whenever he wants a consumption different from his endowment. For one unit of consumption in excess of his endowment, the agent needs to withdraw \(1/k\) units of country-1 good from his financial wealth but, for one unit of endowment not consumed, he can only increase his financial wealth by \(k\) unit of country-2 good. The relevant dynamic budget constraint for his wealth is then nonlinear in consumption, and given by

\[
dX_i(t) = r(t)X_i(t)dt + \pi_{Pi}(t)(\mu(t) - r(t))dt + \pi_{Pi}(t)\sigma(t)dW(t) + k(\epsilon_i(t) - c_i(t))^+ dt - \frac{1}{k}(\epsilon_i(t) - c_i(t))^− dt, \quad i = 2, 3. \quad (4.1)
\]

The second line, nonlinear in \(c_i\), represents the agent’s consumption “expenditures” (over his endowment), expressed in units of country-1 good. Re-formulating the problem as a linear one consisting for the agent in choosing his expenditure process \(e_i\), where \(e_i(t) \equiv k(\epsilon_i(t) - c_i(t))^+ - \frac{1}{k}(\epsilon_i(t) - c_i(t))^−\), leads to a simple solution of agent \(i\)’s problem, using the martingale technology of Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987). His optimal consumption-portfolio policies are reported in Proposition 4.1.
Proposition 4.1. Assume that agent \( i \in \{2, 3\} \) has no domestic security available. Then, his optimal consumption is given by

\[
c_i(t) = \begin{cases} 
I_i(ky_i \xi(t)) & \text{if } \epsilon_i(t) > I_i(ky_i \xi(t)) \\
\epsilon_i(t) & \text{if } I_i(y_i \xi(t)/k) \leq \epsilon_i(t) \leq I_i(ky_i \xi(t)) \\
I_i(y_i \xi(t)/k) & \text{if } \epsilon_i(t) < I_i(y_i \xi(t)/k) 
\end{cases}
\]

where \( y_i > 0 \) satisfies

\[
E \left[ \int_0^T \xi(t) \left\{ k (\epsilon_i(t) - c_i(t))^+ - \frac{1}{k} (\epsilon_i(t) - c_i(t))^\prime \right\} dt \right] = 0. \tag{4.3}
\]

The agent sells consumption and exports (S) in those times and states when his endowment is relatively high or the price of consumption \( \xi \) is relatively high, and buys consumption (B) when his endowment or the price of consumption is relatively low. Due to the shipping costs and the ensuing nonlinearity in the pricing of consumption, agent \( i \) will refrain from trading on an intermediate region of positive measure (N).

For \( i = 2, 3 \), we define agent-specific country-2 state price densities by \( \xi_i^*(t) \equiv u_i'(c_i(t))/y_i \), and (individual-specific) shadow exchange rates by \( p_i(t) \equiv \xi_i^*(t)/\xi(t) \). \( r_i \) and \( \theta_i \) denote \( i \)'s shadow interest rate and market price of risk and are defined by \( d\xi_i^*(t) = -\xi_i^*(t) [r_i(t) dt + \theta_i(t) dW(t)] \).

As will become clear in Section 4.2, \( p_2, p_3 \) are the exchange rates such that the agent would consume as he does here if foreign exchange were indeed allowed. When agents 2 and 3 are homogeneous, for brevity of notation we will refer to their common \( \xi^* \) and \( p_i \) as \( \xi^* \) and \( p \) (since \( \xi^* \), \( p \) do not exist here). (Similarly, in Economy I^*, agent 1 faces a shadow state price \( \xi_1 \), but for notational convenience we will refer to it as \( \xi \).)

4.2. The Case of a Domestic Financial Market

This is the case for agent 1 in Economies I, II and III, and for agents 2 and 3 in Economies II and III. Foreign exchange ("financial shipping") is now possible, at rate \( p \), which by (3.2) ensures that financial shipping is either equivalent or preferable to physical shipping. Hence, agents 2 and 3’s consumption expenditures (above their endowment), expressed in units of country-1 good, are linear in their consumption, and given by \( p(t) (c_i(t) - \epsilon_i(t)) \). Agent 1’s are simply given by \( (c_1(t) - \epsilon_1(t)) \). Accordingly, agents’ dynamic budget constraints, in units of country-1 good, are

\[
dX_1(t) = (\epsilon_1(t) - c_1(t)) dt + \pi M_1(t) dM_1(t)/M(t) + \pi P_1(t) [dP(t) + \delta(t) dt]/P(t) \tag{4.4}
\]

\[
= [r(t) X_1(t) + (\epsilon_1(t) - c_1(t))] dt + [\pi P_1(t) (\mu(t) - r(t))]
\]

\[
+ p(t) \pi M_1(t) (r^*(t) + \mu_p(t) - r(t)) + p(t) \pi P_1(t) (\mu^*(t) + \mu_p(t) + \sigma^*(t) \sigma_p(t) - r(t))] dt
\]

\[
+ [\pi P_1(t) \sigma(t) + p(t) \pi M_1(t) \sigma_p(t) + p(t) \pi P_1(t) \sigma^*(t) \sigma_p(t)] dW(t), \tag{4.5}
\]

and, for agents \( i \in \{2, 3\} \),
\[ dX_i(t) = p(t) (\epsilon_i(t) - c_i(t)) dt + \pi_{M_i}(t)dM(t)/M(t) + \pi_{P_i}(t)[dP(t) + \delta(t)dt] / P(t) \]
\[ + \pi_{M^{*i}}(t)d[p(t)M^{*}(t)] / p(t)M^{*}(t) + \pi_{P^{*i}}(t)[d(p(t)P^{*}(t))] + p(t)\delta^{*}(t)dt] / p(t)P^{*}(t). \]
\[ = [r(t)X_i(t) + p(t) (\epsilon_i(t) - c_i(t))] dt + [\pi_{P_i}(t) (\mu(t) - r(t)) \]
\[ + p(t)\pi_{M^{*i}}(t) (r^{*}(t) + \mu_p(t) - r(t)) + p(t)\pi_{P^{*i}}(t) (\mu^*(t) + \mu_p(t) + \sigma^*(t)\sigma_p(t) - r(t))] dt \]
\[ + [\pi_{P_i}(t)\sigma(t) + p(t)\pi_{M^{*i}}(t)\sigma_p(t) + p(t)\pi_{P^{*i}}(t)\sigma^*(t)\sigma_p(t)] dW(t), \quad (4.7) \]

with \( \pi_{P^{*i}} \equiv 0 \) in Economy II.

Although they enable foreign exchange, the country-2 securities are redundant for risk allocation. We thus introduce the risk-weighted sum \( \Phi^i \) of \( i \)'s holdings in the risky securities:
\[ \Phi^i(t) = \pi_{P_i}(t) + p(t)\pi_{M^{*i}}(t)\frac{\sigma_p(t)}{\sigma(t)} + p(t)\pi_{P^{*i}}(t)\frac{\sigma^*(t)\sigma_p(t)}{\sigma(t)}, \quad (4.8) \]
also interpreted as \( i \)'s composite risk exposure, in that all portfolio strategies leading to the same \( \Phi^i \) yield the same volatility in \( i \)'s dynamic budget constraint. No-arbitrage requires all risky securities to provide identical market prices of risk when expressed in the same unit (country-1 good), as reported by Proposition 4.2.

**Proposition 4.2.** For agent \( i = 1, 2, 3 \)'s optimization to have a solution, it is necessary that:
\[ \frac{\mu(t) - r(t)}{\sigma(t)} = \frac{r^{*}(t) + \mu_p(t) - r(t)}{\sigma_p(t)} = \frac{\mu^*(t) + \mu_p(t) + \sigma^*(t)\sigma_p(t) - r(t)}{\sigma^*(t) + \sigma_p(t)}, \quad \forall t. \quad (4.9) \]
If (4.9) holds, agent \( i \) is indifferent between all \( \pi^i(t) \) leading to the same value for \( \Phi^i(t) \).

In Economy III, where state price densities are defined in both countries, (4.9) implies they are related by \( \xi^*(t) = p(t)\xi(t), \forall t \).

From Proposition 4.2, an agent’s portfolio problem can be reduced to one involving a single, composite risky asset settled in country 1 and with price parameters as \( P \). \( \Phi^i(t) \) can be interpreted as the agent’s amount invested therein. Agents’ dynamic budget constraints in terms of \( \Phi^i \) are:
\[ dX_1(t) = [r(t)X_1(t) + (\epsilon_1(t) - c_1(t))] dt + \Phi_1(t) [(\mu(t) - r(t)) dt + \sigma(t)dW(t)]. \]
\[ dX_i(t) = [r(t)X_i(t) + p(t) (\epsilon_i(t) - c_i(t))] dt + \Phi_i(t) [(\mu(t) - r(t)) dt + \sigma(t)dW(t)], \quad i \in \{2, 3\}. \]

Once reduced to the choice of \( (c_1, \Phi_1) \), agent \( i \)'s optimization problem can be solved using standard martingale techniques (Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987)), leading to the optimal policies in Proposition 4.3, assuming they exist.

**Proposition 4.3.** Agent 1’s optimal consumption in Economies I, II, III is given by
\[ c_1(t) = I_1(y_1\xi(t)), \quad (4.10) \]
where \( y_1 > 0 \) satisfies
\[
E \left[ \int_0^T \xi(t) \left\{ I_1(y_1 \xi(t)) - \epsilon_1(t) \right\} dt \right] = 0.
\]
Agent i’s, \( i \in \{2, 3\} \), optimal consumption in Economies II, III is given by
\[
c_i(t) = I_i(y_i p(t) \xi(t)),
\] (4.11)
where \( y_i > 0 \) satisfies
\[
E \left[ \int_0^T p(t) \xi(t) \left\{ I_i(y_i p(t) \xi(t)) - \epsilon_i(t) \right\} dt \right] = 0.
\] (4.12)

From (4.10), agent 1 sells consumption (S) for high enough endowment or price of consumption, and buys consumption (B) for low enough endowment or price of consumption. His no-trade region shrinks to a knife-edge. Agents 2 and 3’s conditions are similar, only with \( p(t) \) being substituted for \( \xi(t) \). All agents now face a linear problem and, accordingly, trade everywhere except on a space of zero measure. In Economies II and III, agent \( i \in \{2, 3\} \) faces the shadow state price density \( \xi^*_i \), defined by
\[
\xi^*_i(t) = u'_i(c_i(t))/y_i.
\]
From (4.11), \( \xi^*_i(t) = p(t) \xi(t) \) for both agents (not individual-specific). In Economy III where country 2 is endowed with a complete domestic market, \( \xi^*_i \) coincides with the actual country 2 state price density \( \xi^* \) of (3.1).

5. Equilibrium under Intranational Homogeneity

We now assume homogeneity (in preferences and endowments) within country 2, effectively populating it with a single agent \( i = 2 \). Then, the main conclusions of the single period case still hold: under all financial market structures examined, the equilibrium remains Pareto optimal.

5.1. Determination of Equilibrium

Proposition 5.1 provides a method for the determination of equilibrium under intranational homogeneity.

Proposition 5.1. If equilibrium exists, in all Economies I, I*, II and III the amounts of good shipped and real exchange rate are as follows.

When \( y_1 u_2'(e_2(t))/y_2 u'_1(e_1(t)) < k \), country 1 imports and
\[
x^E(t) = 0, \quad x^I(t) > 0 \text{ solves } y_1 u_2'(e_2(t)) - x^I(t)/k = \frac{y_1 u_2'(e_2(t))}{y_2 u'_1(e_1(t))}, \quad p(t) = k; \] (5.1)

when \( k \leq y_1 u_2'(e_2(t))/y_2 u'_1(e_1(t)) \leq 1/k \), countries are in autarky and
\[
x^E(t) = x^I(t) = 0, \quad p(t) = \frac{y_1 u_2'(e_2(t))}{y_2 u'_1(e_1(t))}; \] (5.2)
when \(y_1u_2'(\epsilon_2(t))/y_2u_1'(\epsilon_1(t)) > 1/k\), country 1 exports and
\[
x^I(t) = 0, \quad x^E(t) > 0 \quad \text{solves} \quad \frac{y_1u_2'(\epsilon_2(t) + kx^E(t))}{y_2u_1'(\epsilon_1(t) - x^E(t))} = \frac{1}{k}, \quad p(t) = \frac{1}{k}, \quad (5.3)
\]
where \(y_1/y_2\) solves either agent’s budget constraint, i.e.,
\[
E \left[ \int_0^T u_1'(\epsilon_1(t) - x^E(t) + x^I(t)) \left( x^E(t) - x^I(t) \right) dt \right] = 0. \quad (5.4)
\]

The country-specific state-price-densities and consumption allocations are given by
\[
\xi(t) = u_1'(\epsilon_1(t) - x^E(t) + x^I(t)), \quad \xi^*(t) = p(t)u_1'(\epsilon_1(t) - x^E(t) + x^I(t)), \quad (5.5)
\]
\[
c_1(t) = \epsilon_1(t) - x^E(t) + x^I(t), \quad c_2(t) = \epsilon_2(t) + kx^E(t) - x^I(t)/k. \quad (5.6)
\]

Conversely, if there exist \(x^E, x^I, p, \xi\) and \(\xi^*\) satisfying (5.1)-(5.5), then the associated optimal policies satisfy all market clearing conditions.

As in a frictionless model, the solution of equilibrium reduces to the determination of agents’ relative weight \(y_1/y_2\); Proposition 5.1 provides all other quantities as a function thereof. \(y_1/y_2\) is set so that the flow of international trade satisfies a budget constraint that requires the present value of either country’s exports, net of its imports, to be zero. As in the single period case, the direction of international trade depends on the distribution of the contemporaneous endowment \((u_1'(\epsilon_1(t))/u_2'(\epsilon_2(t)))\); a country will import when it is relatively poor (with respect to the other country), and if the difference is pronounced enough to justify incurring the shipping costs.

From (5.1) and (5.3), we may deduce some sensitivities of the imports and exports. For example, in the region where country 1 imports, the amount of goods arriving there \(x^I(t)\) is decreasing in \(\epsilon_1(t)\) and \(y_1/y_2\) (the relative initial wealth of country 2) and increasing in \(\epsilon_2(t)\), highlighting the risk-sharing role of shipping. \(x^I(t)\) is also increasing in \(k\), i.e., it increases as the shipping costs are alleviated, naturally facilitating trade. The sensitivity to \(k\) of the amount exported by country 2 \((x^I(t)/k)\), however, is indeterminate; as shipping costs are reduced, smaller exports are necessary for the other country to receive the same amount of good, which tends to reduce country 2’s exports even though country 1’s imports unequivocally grow.

As in the single period case, under intranational homogeneity the location of financial markets has no effect on allocations. The intuition of the single period case extends: countries share shipping costs efficiently, and as financial markets are moved away from one country, the relative prices of the available securities change so as to exactly compensate for the shipping costs now incurred. A comparison of Economies I and I* clarifies this point. Take two states \(\omega^E, \omega^I \in \Omega\) where country 1, respectively, exports and imports at some future time \(t'\). The relative prices of the available Arrow-Debreu securities paying in these states can be expressed in Economy I (where the available securities pay-off in country 1) as \(\xi(\omega^E, t')dP(\omega^E|F_t)/\xi(\omega^I, t')dP(\omega^I|F_t)\) and in Economy I* (where they pay-off in country 2) as \(\xi^*(\omega^E, t')dP(\omega^E|F_t)/\xi^*(\omega^I, t')dP(\omega^I|F_t)\).
which from (5.1) and (5.3) equals the ratio in Economy I multiplied by \( k^2 \). This change in price ratio benefits country 2, which is also the country hurt by the move in financial markets. This is the analogue of (2.1) from the single period case.

Proposition 5.1 implies that the equilibrium is “Pareto optimal” (using the term as in Dumas (1992)), in that it can be solved by means of a central planner problem (although marginal rates of substitution are not necessarily equated, due to the shipping costs).

Corollary 5.1. If equilibrium exists, in all Economies I, I*, II and III, the equilibrium allocations solve the following maximization problem:

\[
\begin{align*}
\max_{c_1, c_2} & \quad E \left[ \int_0^T \left( \frac{1}{y_1} u_1(c_1(t)) + \frac{1}{y_2} u_2(c_2(t)) \right) dt \right] \\
\text{subject to} & \quad c_1(t) = e_1(t) - x^E(t) + x^I(t) \\
& \quad c_2(t) = e_2(t) + k x^E(t) - x^I(t)/k \\
& \quad x^E(t) \geq 0, \quad x^I(t) \geq 0, \quad \forall t.
\end{align*}
\] (5.7)

This finding provides support for the body of literature originating in Dumas (1992) (and more extensively reviewed in the Introduction), which employs this technique to determine equilibrium.

As defined in Section 3, \( p \) is not only the shadow relative price of goods, but also the rate at which securities settled in different countries can be exchanged. When securities are only traded in one country (Economies I and I*), \( p \) is a shadow exchange rate: the only rate that, if a foreign exchange contract were available to the agents, would induce identical allocations to our case. It is therefore pinned down uniquely, even in the absence of good trade across countries. Corollary 5.1 additionally allows us to verify that our notion of real exchange rate is consistent with the literature. Denoting the integrand in (5.7) by \( V(t) \) and its partial derivatives with respect to \( c_1(t) \) and \( c_2(t) \) by \( V_{c_1}(t) \) and \( V_{c_2}(t) \) respectively, equations (5.1)-(5.3) reveal that \( p(t) = V_{c_2}(t)/V_{c_1}(t) \), which is the notion of real exchange rate of Dumas (1992) and subsequent related work.

Overall, under intranational homogeneity the conclusions and intuitions of the single period case extend to continuous time; we shall see in Section 6 that this is not the case under intranational heterogeneity.

5.2. Characterization of Equilibrium

This subsection provides characterization of international trade, prices (including the real exchange rate) and consumption dynamics. We present expressions only for the region of no-trade and that where country 1 imports (\( p(t) = k \)), as those for the region where country 1 imports are symmetric. We define agent \( i \)'s absolute risk aversion and prudence coefficients by

\[
A_i(t) \equiv -\frac{u''_i(c_i(t))}{u'_i(c_i(t))}, \quad B_i(t) \equiv -\frac{u'''_i(c_i(t))}{u''_i(c_i(t))}.
\]

For convenience, we introduce a “world representative agent” with utility \( U \) defined by

\[
U(C; p) \equiv \max_{c_1 + pc_2 = C} \frac{1}{y_1} u_1(c_1) + \frac{1}{y_2} u_2(c_2). \quad (5.8)
\]
Identifying $C$ with $\epsilon_1(t) + p(t) \epsilon_2(t)$ and $p$ with $p(t)$ and deriving the first-order conditions for the problem in (5.8) reveals the solution of the representative agent problem to coincide with the equilibrium allocation from Proposition 5.1 ((5.6)). The equilibrium thus effectively coincides with one involving a single (world) representative agent consuming the aggregate world consumption and endowed with the aggregate world endowment, in units of country 1 good. The world representative agent’s absolute risk aversion and prudence coefficients satisfy

$$A(t) \equiv -\frac{U''(\epsilon_1(t) + p(t) \epsilon_2(t); p(t))}{U'(\epsilon_1(t) + p(t) \epsilon_2(t); p(t))} = \frac{1}{\frac{1}{A_1(t)} + p(t) \frac{1}{A_2(t)}};$$

$$B(t) \equiv -\frac{U''(\epsilon_1(t) + p(t) \epsilon_2(t); p(t))}{U'(\epsilon_1(t) + p(t) \epsilon_2(t); p(t))} = \left(\frac{A(t)}{A_1(t)}\right)^2 B_1(t) + p(t) \left(\frac{A(t)}{A_2(t)}\right)^2 B_2(t).$$

(5.9)

(5.10)

For comparison, we also introduce the benchmark economies $a$, without shipping ($k = 0$), and $b$, with zero shipping costs ($k = 1$).

Proposition 5.2 characterizes the dynamics of international trade in the region where country 1 imports.

**Proposition 5.2.** When country 1 imports, the dynamics of the flow of goods arriving in country 1 are given by: $dx^1(t) = \mu_{x^1}(t)dt + \sigma_{x^1}(t)dW(t)$, where

$$\mu_{x^1}(t) = \frac{kA(t)}{A_1(t)A_2(t)} \left(A_2(t)\mu_{\epsilon_2}(t) - A_1(t)\mu_{\epsilon_1}(t)\right) - \frac{1}{2} kA(t)^2 \left(\frac{B_2(t)}{A_2(t)} - \frac{B_1(t)}{A_1(t)}\right) (\sigma_{\epsilon_1}(t) + k\sigma_{\epsilon_2}(t))^2;$$

$$\sigma_{x^1}(t) = \frac{kA(t)}{A_1(t)A_2(t)} \left(A_2(t)\sigma_{\epsilon_2}(t) - A_1(t)\sigma_{\epsilon_1}(t)\right).$$

The volatility in international trade is given by the (risk-tolerance-weighted) difference between the volatilities of the countries’ endowments. Similarly, the growth in international trade is driven by the weighted difference in expected growth rates. The growth and volatility of international trade are impacted more by the endowment growth and volatility of the more risk averse agent; consumption smoothing is more valuable to him so flows of goods have to track the fluctuations in his endowment more closely. In a similar fashion, whether exports are pro- or counter-cyclical (i.e., $\sigma_{x^2} > 0$ or $< 0$) depends on which country is more affected by endowment risk (i.e., has the higher $A_1\sigma_{\epsilon_i}$). If this is the case of the importing country, then its exports are counter-cyclical because trade is driven primarily by the need to counterbalance adverse fluctuations in its endowment. The above discussion is equally valid in the economy with no shipping costs (benchmark $b$). What we observe different here is that: growth and volatility of trade are weighed by $k$ to reflect the reduction of trade due to the shipping cost; and when aggregating countries’ endowment volatilities, the exporting country’s (country 2’s) volatility is scaled down.

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6Unlike in frictionless economies (e.g., Karatzas, Lehoczky and Shreve (1990)), solution of this world representative agent’s problem is not sufficient for the determination of equilibrium because of the presence of the endogeneous $p(t)$. Clearing in the aggregate world good (whose supply is given by $\epsilon_1 + p\epsilon_2$) is a necessary but not sufficient condition for good clearing in the individual countries.

7This assumes that countries’ endowments are positively correlated, i.e., $\sigma_{\epsilon_1}\sigma_{\epsilon_2} > 0$. 

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by the factor $k$. This captures the fact that in this region marginal changes in country 2 endowment are worth less (by a factor of $k$) in country 1 (the country that is receiving the flow of goods $x'$) terms, since such goods need to be shipped to country 1.

Proposition 5.3 provides expressions for the country-specific price parameters and consumption volatilities.\footnote{These shadow prices coincide with the prices of the corresponding securities when these are available.}

**Proposition 5.3.** The equilibrium country-specific interest rate and market prices of risk are:

- when country 1 imports:
  \[
  r_1(t) = r_2(t) = A(t) (\mu_1(t) + k\mu_2(t)) - \frac{A(t)B(t)}{2} (\sigma_{\epsilon_1}(t) + k\sigma_{\epsilon_2}(t))^2 ,
  \]
  \[
  \theta_1(t) = \theta_2(t) = A(t) (\sigma_{\epsilon_1}(t) + k\sigma_{\epsilon_2}(t));
  \]
- when no trade occurs across countries:
  \[
  r_i(t) = A_i(t)\mu_{\epsilon_i}(t) - \frac{A_i(t)B_i(t)}{2} \sigma_{\epsilon_i}(t)^2, \quad i \in \{1, 2\},
  \]
  \[
  \theta_i(t) = A_i(t)\sigma_{\epsilon_i}(t), \quad i \in \{1, 2\}.
  \]

The equilibrium consumption dynamics are:

\[
dc_i(t) = \mu_{c_i}(t)dt + \sigma_{c_i}(t)dW(t); \quad \sigma_{c_i}(t) = \theta_i(t)/A_i(t).
\]

In the trade regions, interest rates and market prices of risk are equated across countries. The interest rate is positively related to the growth in endowment of the world representative agent, normalized by the world representative agent’s absolute risk tolerance. Also the interest rate is negatively related (for decreasing absolute risk aversion agents) to the world representative agent’s endowment risk, weighted by his risk aversion and prudence. The market price of risk is given by the world representative agent’s endowment risk weighted by his risk aversion. The economy then indeed behaves as if populated by a single world agent endowed with the two goods, with a relative price equal to $k$. Since agents face identical market prices of risk, the trade allows consumption risk to be shared in the same proportions as in the frictionless economy (benchmark $b$). The expressions reveal the endowment of the exporting country to have a smaller impact on equilibrium prices (an effect more pronounced when shipping costs are higher); the contribution of country 2’s risk tolerance is scaled up by $k$ and that of his endowment volatility is scaled down by $k$. Expressions in the trade region are otherwise identical to the economy ($b$) with no shipping costs. All the expressions of the no-trade region in our economy are naturally identical to those in the no-trade benchmark $a$. The individual countries’ interest rates and market prices of risk are not equated and are driven by the individual country’s endowment, risk aversion and prudence.

Corollary 5.2 presents the real exchange rate dynamics.

**Corollary 5.2.** The exchange rate dynamics are as follows.

When country 1 imports, $p(t) = k$ and $\mu_p(t) = \sigma_p(t) = 0$. 
When no trade occurs across countries, \( p(t) = y_1 u_2'(\epsilon_2(t)) / y_2 u_1'(\epsilon_1(t)) \) and:

\[
\begin{align*}
\mu_p(t) &= (A_1(t)\mu_{\epsilon_1}(t) - A_2(t)\mu_{\epsilon_2}(t)) - \frac{1}{2} \left( A_1(t)B_1(t)\sigma_{\epsilon_1}(t)^2 - A_2(t)B_2(t)\sigma_{\epsilon_2}(t)^2 \right) \\
&\quad + A_1(t)^2\sigma_{\epsilon_1}(t)^2 - A_1(t)A_2(t)\sigma_{\epsilon_1}(t)\sigma_{\epsilon_2}(t), \\
\sigma_p(t) &= A_1(t)\sigma_{\epsilon_1}(t) - A_2(t)\sigma_{\epsilon_2}(t).
\end{align*}
\]

(5.15)

(5.16)

The real exchange rate reflects the imbalance in marginal utility across countries. It is thus driven primarily by the differential in endowment growth parameters, normalized by the risk aversions, between the two countries. The first line in (5.15) equals the interest rate differential \( r_1 - r_2 \), which is thus established to be a biased predictor of the evolution of the exchange rate. These two quantities are among the main objects of interest in Dumas (1992). Comparisons with his work are difficult, because (in a production economy) he employs a more specialized model and thus obtains sharper results. Nevertheless, to ease comparison with his work (equation (23)), we rewrite the expected (log-) growth of the real exchange rate as

\[
\frac{E[\ln p(t)]}{dt} = (r_1(t) - r_2(t)) + \frac{1}{2} (A_1(t)\sigma_{\epsilon_1}(t) + A_2(t)\sigma_{\epsilon_2}(t)) \sigma_p(t).
\]

The difference between the expected log-growth of the exchange rate and the interest rate differential is revealed to be proportional to the exchange rate volatility and the “aggregate” consumption risk times risk aversion, and can thus be interpreted as a foreign exchange risk premium.

6. Equilibrium under Intranational Heterogeneity

We now return to the general case of heterogeneity within country 2. Then, the conclusions of Section 5 do not hold; equilibria differ depending on the financial market structure. We examine separately the case of no domestic financial market within country 2 (Section 6.1) and the case where at least one security is settled in country 2 (Section 6.2).

6.1. The Case of no Domestic Financial Market within Country 2

This is Economy I. As we shall see, it may now be necessary to distinguish country 1’s exports and imports depending on which country-2 agent they go to/originate from. Thus, we introduce the notations \( x_i^e, x_i' \), for \( i \in \{2, 3\} \), to denote, respectively, country 1’s exports purchased by agent \( i \) and its imports purchased from agent \( i \). We have: \( x^e(t) = x_2^e(t) + x_3^e(t) \) and \( x'(t) = x_2'(t) + x_3'(t) \), \( \forall t \), and agents’ consumptions are given by: \( c_1(t) = \epsilon_1(t) - x_2^e(t) + x_2'(t) + x_3'(t), c_2(t) = \epsilon_2(t) + kx_2^e(t) - x_2'(t)/k \) and \( c_3(t) = \epsilon_3(t) + kx_3^e(t) - x_3'(t)/k \). Since they have no domestic financial market, the only way for agents 2 and 3 to trade (even with each other) consists in importing or exporting.

Proposition 6.1 provides a procedure for the determination of equilibrium. We distinguish nine cases dependent upon the shipping situation (S, N, B) of the nonlinear agents (2 and 3).
All regions not presented (BS, NS, NB) are “mirror images” of regions presented (SB, SN, BN respectively).

**Proposition 6.1.** If equilibrium exists, in Economy I the amounts of good shipped and individual-specific real exchange rates are as follows:

(3S) agents 2 and 3 export when

\[
\frac{y_1}{y_2} \frac{u'_2(e_2(t))}{u'_1(e_1(t) + x'_3(t))} < k \quad \text{and} \quad \frac{y_1}{y_3} \frac{u'_3(e_3(t))}{u'_1(e_1(t) + x'_3(t))} < k, \tag{6.1}
\]

and we have: \(p_2(t) = p_3(t) = k, \ x'_2(t) = x'_3(t) = 0\), and \(x'_2(t), x'_3(t) > 0\) satisfy

\[
y_1 \frac{u'_2(e_2(t) - x'_2(t)/k)}{y_2 u'_1(e_1(t) + x'_2(t) + x'_3(t))} = k \quad \text{and} \quad y_1 \frac{u'_3(e_3(t) - x'_3(t)/k)}{y_3 u'_1(e_1(t) + x'_2(t) + x'_3(t))} = k; \tag{6.2}
\]

(3N) agents 2 and 3 do not trade when

\[
k \leq \frac{y_1}{y_2} \frac{u'_2(e_2(t))}{u'_1(e_1(t))} \leq \frac{1}{k} \quad \text{and} \quad k \leq \frac{y_1}{y_3} \frac{u'_3(e_3(t))}{u'_1(e_1(t))} \leq \frac{1}{k},
\]

and we have: \(x'_2(t) = x'_3(t) = x'_4(t) = 0\), \(p_2(t) = \frac{y_1}{y_2} \frac{u'_2(e_2(t))}{u'_1(e_1(t))} \) and \(p_3(t) = \frac{y_1}{y_3} \frac{u'_3(e_3(t))}{u'_1(e_1(t))}\);

(3B) agents 2 and 3 import when

\[
y_1 \frac{u'_2(e_2(t))}{y_2 u'_1(e_1(t) - x'_3(t))} > \frac{1}{k} \quad \text{and} \quad y_1 \frac{u'_3(e_3(t))}{y_3 u'_1(e_1(t) - x'_2(t))} > \frac{1}{k}, \tag{6.3}
\]

and we have: \(p_2(t) = p_3(t) = 1/k, \ x'_2(t) = x'_4(t) = 0\), and \(x'_2(t), x'_3(t) > 0\) satisfy

\[
y_1 \frac{u'_2(e_2(t) + kx'_2(t))}{y_2 u'_1(e_1(t) - x'_2(t) - x'_3(t))} = \frac{1}{k} \quad \text{and} \quad y_1 \frac{u'_3(e_3(t) + kx'_3(t))}{y_3 u'_1(e_1(t) - x'_2(t) - x'_3(t))} = \frac{1}{k}; \tag{6.4}
\]

(3S) agent 2 exports and agent 3 imports when

\[
y_1 \frac{u'_2(e_2(t))}{y_2 u'_1(e_1(t) - x'_3(t))} < k \quad \text{and} \quad y_1 \frac{u'_3(e_3(t))}{y_3 u'_1(e_1(t) + x'_2(t))} > \frac{1}{k},
\]

and we have: \(p_2(t) = k, p_3(t) = 1/k, x'_2(t) = x'_3(t) = 0\), and \(x'_2(t), x'_3(t) > 0\) satisfy

\[
y_1 \frac{u'_2(e_2(t) - x'_2(t)/k)}{y_2 u'_1(e_1(t) + x'_2(t) - x'_3(t))} = k \quad \text{and} \quad y_1 \frac{u'_3(e_3(t) + kx'_3(t))}{y_3 u'_1(e_1(t) + x'_2(t) - x'_3(t))} = \frac{1}{k}; \tag{6.4}
\]

(3N) agent 2 exports and agent 3 does not trade when

\[
y_1 \frac{y_2 u'_2(e_2(t))}{y_2 u'_1(e_1(t))} < k \quad \text{and} \quad k \leq \frac{y_1}{y_3} \frac{u'_3(e_3(t))}{u'_1(e_1(t) + x'_2(t))} \leq \frac{1}{k},
\]

and we have: \(p_2(t) = k, x'_2(t) = x'_3(t) = x'_4(t) = 0\), and \(x'_2(t) > 0\), \(p_3(t)\) satisfies

\[
y_1 \frac{u'_2(e_2(t) - x'_2(t)/k)}{y_2 u'_1(e_1(t) + x'_2(t))} = k \quad \text{and} \quad p_3(t) = \frac{y_1}{y_3} \frac{u'_3(e_3(t))}{u'_1(e_1(t) + x'_2(t))}; \tag{6.5}
\]
(BN) agent 2 imports and agent 3 does not trade when
\[ \frac{y_1 u'_2(\epsilon_2(t))}{y_2 u'_1(\epsilon_1(t))} > \frac{1}{k} \quad \text{and} \quad k \leq \frac{y_1}{y_3} \frac{u'_3(\epsilon_3(t))}{u'_1(\epsilon_1(t) - x^e_2(t))} \leq \frac{1}{k}, \]

and we have: \( p_2(t) = 1/k, \) \( x^e_2(t) = x^e_3(t) = x^e_4(t) = 0, \) and \( x^e_5(t) > 0, \) \( p_3(t) \) satisfy
\[ \frac{y_1 u'_2(\epsilon_2(t) + x^e_5(t))}{y_2 u'_1(\epsilon_1(t) - x^e_2(t))} = \frac{1}{k} \quad \text{and} \quad p_3(t) = \frac{y_1}{y_3} \frac{u'_3(\epsilon_3(t))}{u'_1(\epsilon_1(t) - x^e_2(t))}. \] (6.6)

where \( y_1/y_2, \) \( y_1/y_3 \) solve any two of the agents’ budget constraints, i.e.,
\[ E \left[ \int_0^T u'_2(\epsilon_2(t) + kx^e_2(t) - x^e_2(t)/k) (kx^e_2(t) - x^e_2(t)/k) \right] = 0; \]
\[ E \left[ \int_0^T u'_3(\epsilon_3(t) + kx^e_3(t) - x^e_3(t)/k) (kx^e_3(t) - x^e_3(t)/k) \right] = 0. \]

The agent-specific state price densities and consumption allocations are given by
\[
\xi(t) = u'_1(\epsilon_1(t) - x^e_2(t) - x^e_3(t) + x^e_4(t) + x^e_5(t)), \quad c_1(t) = \epsilon_1(t) - x^e_2(t) - x^e_3(t) + x^e_4(t) + x^e_5(t), \\
\xi^*_2(t) = p_2(t) u'_1(\epsilon_1(t) - x^e_2(t) - x^e_3(t) + x^e_4(t) + x^e_5(t)), \quad c_2(t) = \epsilon_2(t) + kx^e_2(t) - x^e_2(t)/k, \\
\xi^*_3(t) = p_3(t) u'_1(\epsilon_1(t) - x^e_2(t) - x^e_3(t) + x^e_4(t) + x^e_5(t)), \quad c_3(t) = \epsilon_3(t) + kx^e_3(t) - x^e_3(t)/k. \]
(6.7)

Conversely, if there exist \( x^e_2, x^e_3, x^e_4, p_2, p_3, \) \( \xi^*_2 \) and \( \xi^*_3 \) satisfying (6.1)-(6.7), then the associated optimal policies satisfy all market clearing conditions.

Again, as in a frictionless economy, the solution of equilibrium can be reduced to the determination of agents’ relative weights \( y_1/y_2, \) \( y_1/y_3. \) Nevertheless, in this setup one cannot solve for the equilibrium as in Dumas (1992) and subsequent literature, by considering a central planner with constant weights for each country: this is because shipping costs now prevent efficient risk sharing within country 2. Whenever agents 2 and 3 are heterogeneous enough to make different decisions as to whether to buy or sell consumption, they face effectively different state prices (e.g., in SB, \( \xi^*_1 = k \xi^*_2 \) and \( \xi^*_2 = \xi^*_3/\xi^*_2 \)) and so the allocation within country 2 is not Pareto optimal.

As in the single period model, this reflects both the “wasteful shipping” in cases SB and BS and, more generally, agents 2 and 3 failing to equate their marginal rates of substitution in all cases but SS and BB. In this economy, the situation is essentially similar to the single period analysis.\(^9\)

In spite of the properties of equilibrium being quite different from the intranational homogeneity case, in most situations the characterization of equilibrium can be easily adapted from Section 5. In SS and BB, the allocation within country 2 is Pareto optimal, so we can substitute

\(^9\)The implication that goods are simultaneously exported and imported (in regions SB and BS) may seem counter-intuitive. Our model, however, could be interpreted as a proxy for a multi-good model, where exports and imports consist of different goods, while imperfect domestic risk-sharing within country 2 (between agents endowed with different goods) leads to aggregate quantities as in our model. Another case of relevance for the results in this subsection is that of a 3-country model, where each agent represents a different country and only one country is endowed with financial markets. Characterization would then be identical to our case.
country 2 as a single agent as in Section 5. In SN, NS, BN or NB, the expressions for the agent not trading are as those for an agent not trading in Section 5, while the expressions for the trading agent are as those for agent 2. In NN, the expressions for all agents are those provided for autarky in Section 5.

The only regions where the expressions are substantially new in the three agent model are SB and BS. Proposition 6.2 reports the dynamics of international trade and the individual specific price parameters and consumption volatilities in region SB. We may introduce a “world representative agent” with utility defined by

\[ U(C; p_2, p_3) \equiv \max_{c_1 + p_2 c_2 + p_3 c_3 = C} \frac{1}{y_1} u_1(c_1) + \frac{1}{y_2} u_2(c_2) + \frac{1}{y_3} u_3(c_3). \]  

(6.8)

Identifying \( C \) with \( \epsilon_1(t) + p_2(t) \epsilon_2(t) + p_3(t) \epsilon_3(t) \) and \( p_2 \) and \( p_3 \) with \( p_2(t) \) and \( p_3(t) \) respectively, we may verify that the allocations from the representative agent problem (6.8) coincide with the equilibrium ones. The world representative agent’s absolute risk aversion and prudence coefficients satisfy

\[ A(t) \equiv \frac{-U''(\epsilon_1(t) + p_2(t) \epsilon_2(t) + p_3(t) \epsilon_3(t); p_2(t), p_3(t))}{U'(\epsilon_1(t) + p_2(t) \epsilon_2(t) + p_3(t) \epsilon_3(t); p_2(t), p_3(t))} = \frac{1}{A_1(t) + p_2(t) A_2(t) + p_3(t) A_3(t)}; \]

\[ B(t) \equiv \frac{-U''(\epsilon_1(t) + p_2(t) \epsilon_2(t) + p_3(t) \epsilon_3(t); p_2(t), p_3(t))}{U'(\epsilon_1(t) + p_2(t) \epsilon_2(t) + p_3(t) \epsilon_3(t); p_2(t), p_3(t))} = \left( \frac{A(t)}{A_1(t)} \right)^{2} B_1(t) + p_2(t) \left( \frac{A(t)}{A_2(t)} \right)^{2} B_2(t) + p_3(t) \left( \frac{A(t)}{A_3(t)} \right)^{2} B_3(t). \]

**Proposition 6.2.** If equilibrium exists and region SB occurs, the flow of goods arriving in country 1 (from agent 2), and leaving from country 1 (to agent 3), respectively, follow dynamics:

\[ dx_2(t) = \mu_x(t) dt + \sigma_x(t) dW(t) \]

and

\[ dx_3(t) = \mu_x(t) dt + \sigma_x(t) dW(t), \]

where

\[ \mu_x(t) = \frac{A(t)}{A_1(t) A_2(t) A_3(t)} \left\{ k A_2(t) [A_2(t) \mu_2(t) - A_1(t) \mu_1(t)] + A_1(t) [A_2(t) \mu_2(t) - A_3(t) \mu_3(t)] \right\}; \]

\[ \sigma_x(t) = \frac{A(t)}{A_1(t) A_2(t) A_3(t)} \left\{ k A_2(t) \sigma_2(t) - A_1(t) \sigma_1(t) + A_1(t) [A_2(t) \sigma_2(t) - A_3(t) \sigma_3(t)] \right\}; \]

\[ \mu_x(t) = \frac{A(t)}{A_1(t) A_2(t) A_3(t)} \left\{ \frac{A_2(t)}{k} [A_1(t) \mu_1(t) - A_3(t) \mu_3(t)] + A_1(t) [A_2(t) \mu_2(t) - A_3(t) \mu_3(t)] \right\}; \]

\[ \sigma_x(t) = \frac{A(t)}{A_1(t) A_2(t) A_3(t)} \left\{ \frac{A_2(t)}{k} [A_1(t) \sigma_1(t) - A_3(t) \sigma_3(t)] + A_1(t) [A_2(t) \sigma_2(t) - A_3(t) \sigma_3(t)] \right\}. \]

The individual-specific interest rates and market prices of risk are:

\[ \theta_1(t) = \theta_2(t) = \theta_3(t) = A(t) \left( \sigma_1(t) + k \sigma_2(t) + \frac{1}{k} \sigma_3(t) \right); \]
When $y$ follows.

exists, in all Economies I*, II, III the amounts of good shipped and real exchange rate are as Proposition 6.3. Assume there exists a financial market in country 2. Then, if equilibrium summarizes the resulting construction of equilibrium.

fashion) for the two agents in country 2, and then apply the results of Section 5. Proposition 6.3

The equilibrium consumption dynamics are: $dc_i(t) = \mu_{c_i}(t)dt + \sigma_{c_i}(t)dW(t); \sigma_{c_i}(t) = \theta_i(t)/A_i(t)$.

The shipping dynamics still reflect, as in the intranational homogeneity case, the imbalance in endowment across agents. The expressions are more complicated than (??)-(??), however, because shipments are needed not only for trade between countries, but also within country 2. Accordingly, the shipment dynamics are driven by the (risk-aversion weighted) differences in endowment growth and volatility between each of agents 2 and 3 individually and agent 1 (international trade) on the one hand, and between agents 2 and 3 (country-2 intranational trade) on the other hand. Each of agents 2 and 3’s trade with agent 1 is determined separately, because country 2 cannot be aggregated into a single agent. Price parameter expressions, however, are still as in the “world” representative agent economy.

6.2. The Case of a Domestic Financial Market in Both Countries

We now assume that there exists at least one security settled in country 2 (Economies I*, II and III). Then, from Section 4.2, agents 2 and 3 face a linear problem with a homogeneous state price density $\xi^* = p\xi$. Hence, we can substitute a representative agent (defined in a standard fashion) for the two agents in country 2, and then apply the results of Section 5. Proposition 6.3 summarizes the resulting construction of equilibrium.

Proposition 6.3. Assume there exists a financial market in country 2. Then, if equilibrium exists, in all Economies I*, II, III the amounts of good shipped and real exchange rate are as follows.

When $y_1u''(\epsilon_2(t) + \epsilon_3(t))/y_2u_1'(\epsilon_1(t)) < k$, country 1 imports and

$$x_E(t) = 0, \quad x_I(t) > 0 \text{ solves } \frac{y_1u''(\epsilon_2(t) + \epsilon_3(t)) - x_I(t)/k}{u_1'(\epsilon_1(t) + x_I(t))} = k, \quad p(t) = k. \quad (6.9)$$

When $k \le y_1u''(\epsilon_2(t) + \epsilon_3(t))/y_2u_1'(\epsilon_1(t)) \le 1/k$, there is no international trade and

$$x_E(t) = x_I(t) = 0, \quad p(t) = \frac{y_1u''(\epsilon_2(t) + \epsilon_3(t))}{y_2u_1'(\epsilon_1(t))}. $$

When $y_1u''(\epsilon_2(t) + \epsilon_3(t))/y_2u_1'(\epsilon_1(t)) > 1/k$, country 1 exports and

$$x_I(t) = 0, \quad x_E(t) > 0 \text{ solves } \frac{y_1u''(\epsilon_2(t) + \epsilon_3(t) + kx_E(t))}{u_1'(\epsilon_1(t) - x_E(t))} = \frac{1}{k}, \quad p(t) = \frac{1}{k},$$

where $u^*(\cdot)$ is the country 2 representative agent utility function defined by

$$u^*(c) \equiv \max_{c_2 + c_3 = c} u_2(c_2) + \frac{y_2}{y_3} u_3(c_3), \quad (6.10)$$

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Conversely, if there exist policies satisfying all market clearing conditions.

\[
E \left[ \int_0^T u'_1(\epsilon_1(t) - x_E(t) + x_I(t))(x_E(t) - x_I(t)) dt \right] = 0; \tag{6.11}
\]
\[
E \left[ \int_0^T p(t)u'_1(\epsilon_1(t) - x_E(t) + x_I(t)) \left( I_2(y_2p(t)u'_1(\epsilon_1(t) - x_E(t) + x_I(t))) - \epsilon_2(t) \right) dt \right] = 0. \tag{6.12}
\]

The country-specific state price densities and consumption allocations are

\[
\begin{align*}
\xi(t) &= u'_1(\epsilon_1(t) - x_E(t) + x_I(t)), \\
\xi^*(t) &= p(t)u'_1(\epsilon_1(t) - x_E(t) + x_I(t)), \\
c_1(t) &= \epsilon_1(t) - x_E(t) + x_I(t), \\
c_2(t) &= I_2(y_2\xi^*(t)), \\
c_3(t) &= I_3(y_3\xi^*(t)). \tag{6.13}
\end{align*}
\]

Conversely, if there exist \(x_E, x_I, p, \xi \text{ and } \xi^* \) satisfying (6.9)-(6.13), then the associated optimal policies satisfy all market clearing conditions.

Once the relative weights \((y_1/y_2, y_2/y_3)\) have been determined, the equilibrium can be solved for in two independent stages: the determination of international trade across countries and that of trade within country 2. The conditions for international trade and expressions for agent 1’s consumption are as in the two-agent model of Section 5. The determination of trade within country 2, on the other hand, is as in a frictionless model; consequently, for \(i \in \{2, 3\}\), whether agent \(i\) buys consumption, does not trade or sells consumption, depends on whether, respectively, \(\epsilon_i(t) < I_i(y_ip(t)\xi(t))\), \(\epsilon_i(t) = I_i(y_ip(t)\xi(t))\) or \(\epsilon_i(t) > I_i(y_ip(t)\xi(t))\).

In Economy II, where the country 2 domestic financial market is still incomplete, this important simplification in the determination of equilibrium relies on the availability of foreign exchange: foreign exchange allows agents 2 and 3 to ship financially and, by so doing, to trade with each other without incurring shipping costs. A Pareto optimal allocation results within country 2, as does the possibility to substitute a country-2 representative agent. The availability of financial shipping ensures that no wasteful shipping takes place. This is because, when financial shipping opportunities are present, an agent will import or export physically only if it is not worse for him to do so rather than ship financially: e.g., agent 1 will export (physically) only if \(p(t) = 1/k\) and import only if \(p(t) = k\); otherwise, he would get a more favorable rate by doing financial shipping. Similarly, agents 2 and 3 will export only if \(p(t) = k\) and import only if \(p(t) = 1/k\). Hence, there can be a shipment of goods only in one direction simultaneously, because the foreign exchange rate \(p(t)\) is common to all agents. We note that, somewhat surprisingly, foreign exchange only facilitates risk-sharing at the intranational level.

Since, in our model, foreign exchange involves trade in securities (as described in Section 3.2), one might worry that this could interact with agents’ portfolio strategies and perturb risk-sharing. This is not the case because any financial transaction takes place instantly: the agent purchases securities, immediately exchanges these for other securities and then exchanges these securities for goods. So, at the outset of the transaction, the agents’ portfolio holdings are unaffected.
Since a representative agent can now be substituted for the two agents in country 2, the characterization of economic quantities is as in the two-agent model of Section 5. The only modification is to replace agent 2’s endowment $\epsilon_2$ by country 2’s aggregate endowment $\epsilon_2 + \epsilon_3$, and agent 2’s risk aversion and prudence by the country 2 representative agent’s risk aversion and prudence, defined as follows:

$$A^*(t) \equiv \frac{1}{A_2(t) + A_3(t)}, \quad B^*(t) \equiv \left(\frac{A^*(t)}{A_2(t)}\right)^2 B_2(t) + \left(\frac{A^*(t)}{A_3(t)}\right)^2 B_3(t).$$

Unlike the world representative agent defined in Sections 5 and 6.1, both agents contribute symmetrically to this representative agent. Financial shipping and the ensuing linearization of agents’ optimization problems make this stronger type of aggregation possible. Our analysis thus provides a justification for the procedure (common in the international finance literature: Dumas (1992), etc...) of substituting a representative agent for each country, even when countries are not endowed with complete domestic financial markets. This entails no strong assumption on preferences (unlike, e.g., in the example given by Obstfeld and Rogoff (1996, Section 5.2.2), where constant relative risk aversion preferences are assumed), and only relatively mild (in light of the recent evolution of actual financial markets) requirements on financial structure: the presence of securities with a positive value in each country and the absence of restrictions to international capital flows are sufficient.

**Remark 6.1 (Extensions and Ramifications).** Our model could be adapted easily to incorporate:

(i) **tariffs.** Our “shipping costs” would only require to be returned to the economy, as endowments to the agents, to be able to be interpreted as tariffs. The solution and characterization of equilibrium would be similar to the case examined in the paper, with each agent’s endowment being modified to account for the redistribution of tariff proceeds.

(ii) **positive net supply risky securities.** Assuming the agents to be endowed, rather than the endowment processes $\epsilon_1, \epsilon_2, \epsilon_3$ with positive net supply securities paying (in country 1 and 2 respectively) the dividend processes $\delta$ and $\delta^*$ would entail no significant changes in the characterization of equilibrium, with endowments replaced by dividends in the expressions, and the agents’ static budget constraints ((4.3), (4.12)) adjusted appropriately. For example, agent 1’s (assumed to be endowed with $\epsilon_1$ unit(s) of the country 1 security and $\epsilon_1^*$ unit(s) of the country 2 security; $\epsilon_1^* = 0$ if no country 2 security is available) static budget constraint would become:

$$E\left[\int_0^T \xi(t)c_1(t)dt\right] \leq \epsilon_1E\left[\int_0^T \xi(t)\delta(t)dt\right] + \epsilon_1^*E\left[\int_0^T \xi(t)p(t)\delta^*(t)dt\right]$$

(and analogously for agents 2 and 3). Comparison across economies would become more difficult, however, because modifying the financial market structure would affect the endowments.
(iii) higher-dimensional uncertainty. Our approach could be easily adapted to the case where the underlying Brownian motion has a dimension higher than one. Then, additional risky securities would be needed to complete markets, and characterizations would become more cumbersome (with, in particular, multi-dimensional market prices of risk), but our main point on the role of additional securities (over those needed for international market completeness) and foreign exchange would not be affected.\footnote{Adopting multi-dimensional uncertainty (with possibly country-specific shocks) would be necessary to study some important issues, such as international portfolio choice and financial flows (as, in our case, domestic and foreign securities are redundant).}

7. Conclusion

This paper examines a number of economies with an internationally segmented good market, distinguished by their assumptions on the structure of financial markets. It is shown how new securities (redundant in that they do not enlarge the space of attainable payoffs) alleviate the burden on risk-sharing imposed by the segmentation. This is of importance given the recent evolution of the international financial system, with financial markets becoming more complete while markets for goods remained quite segmented. Interestingly, the mechanism through which the alleviation occurs differs depending on the modelling setup: in a single-period framework, the new securities are used for risk-sharing; in a multiperiod model, however, they are used also for the foreign exchange of goods, allowing some circumvention of the shipping costs that generate the good market segmentation (but only facilitating risk-sharing within countries, and not across countries). Accordingly, the requirements (in terms of financial market structure) for perfect risk-sharing within countries are much less stringent in a multiperiod setup than in a single-period framework. Our work thus provides support for an important part of the recent international finance literature, that substitutes a representative agent for each country, effectively assuming perfect risk-sharing within countries, as the conditions for this are a reasonable fit to the stylized facts of the current international financial system. We can also offer some partial insight on: why in the past few decades the growth in international financial flows far outpaced the increase in good trading, because we show that in an economy with liberalized capital flows, these may have replaced some good trading for risk-sharing purposes; and why studies on benefits of international financial integration found these to be so small, because they neglected the benefits of integration for risk-sharing between heterogeneous agents from the same country. We thus show heterogeneity within a country to be a possibly important fact to incorporate into international finance models, that might help explain other puzzles.
Appendix A: The Single-Period Model

This Appendix provides more detail on the single-period model of Section 2.

Optimal Policies. We denote agent i’s security holdings (in number of units) \( a_i = (a_{i1}, a_{i2}, a_{i1*}, a_{i2*}) \).

Agent 1’s optimization problem consists in choosing his portfolio so as to:

\[
\begin{align*}
\max_{a_1} & \quad \log c_{a1} + \log c_{b1} \\
\text{subject to} & \quad c_{a1} = \epsilon_{a1} + a_{a1} + k(a_{a1*}) + \frac{1}{k}(a_{a1})^{-} \\
& \quad c_{b1} = \epsilon_{b1} + a_{b1} + k(a_{b1*}) + \frac{1}{k}(a_{b1})^{-} \\
& \quad p_a a_{a1} + p_{a*} a_{a1*} + p_b a_{b1} + p_{b*} a_{b1*} = 0,
\end{align*}
\]

(A.1)

where \( a_{a1*} = a_{b1*} = 0 \) in Economy I, \( a_{a1} = a_{b1} = 0 \) in Economy I*, \( a_{b1} = 0 \) in Economy II. For \( i \in \{2, 3\} \), agent i’s optimization problem consists in choosing his portfolio so as to:

\[
\begin{align*}
\max_{a_i} & \quad \log c_{ai} + \log c_{bi} \\
\text{subject to} & \quad c_{ai} = \epsilon_{ai} + a_{ai} + k(a_{ai*}) + \frac{1}{k}(a_{ai})^{-} \\
& \quad c_{bi} = \epsilon_{bi} + a_{bi} + k(a_{bi*}) + \frac{1}{k}(a_{bi})^{-} \\
& \quad p_a a_{ai} + p_{a*} a_{ai*} + p_b a_{bi} + p_{b*} a_{bi*} = 0,
\end{align*}
\]

(A.2)

where \( a_{ai*} = a_{bi*} = 0 \) in Economy I, \( a_{ai} = a_{bi} = 0 \) in Economy I*, \( a_{bi*} = 0 \) in Economy II.

Agent i’s problem can be solved in two stages: first, solve for the optimal distribution of his consumption across states a and b; second (in the presence of redundant securities, i.e., in Economies II and III), pin down the sharing between the redundant securities. Tables I and II provide the solution to the first stage of the optimization problem for agent 1, and agents 2 and 3, respectively, driven by the distribution of their endowment across states. For clarity, we denote an agent’s situation by S, N, or B depending on whether he sells the consumption good in state a (\( c_{ai} < \epsilon_{ai} \)) (S), buys the consumption good in state a (i.e., \( c_{ai} > \epsilon_{ai} \)) (B), or does not trade (\( c_{ai} = \epsilon_{ai} \)) (N).

Table I: Agent 1’s optimal consumption policies

<table>
<thead>
<tr>
<th></th>
<th>( \epsilon_{a1}/\epsilon_{b1} )</th>
<th>( c_{a1} - \epsilon_{a1} )</th>
<th>( c_{b1} - \epsilon_{b1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel (a): No</td>
<td>S</td>
<td>( \frac{\epsilon_{a1}}{\epsilon_{b1}} \geq \frac{1}{k^2 \rho_a} )</td>
<td>( \frac{\epsilon_{a1}}{\epsilon_{b1}} \geq \frac{1}{k^2 \rho_a} )</td>
</tr>
<tr>
<td>domestic financial market  (Economy I*)</td>
<td>N</td>
<td>( \frac{\epsilon_{a1}}{\epsilon_{b1}} &lt; \frac{1}{k^2 \rho_a} )</td>
<td>( \frac{\epsilon_{a1}}{\epsilon_{b1}} &lt; \frac{1}{k^2 \rho_a} )</td>
</tr>
<tr>
<td>B</td>
<td>( \frac{\epsilon_{a1}}{\epsilon_{b1}} &lt; \frac{1}{k^2 \rho_a} )</td>
<td>( \frac{\epsilon_{a1}}{\epsilon_{b1}} &lt; \frac{1}{k^2 \rho_a} )</td>
<td>( \frac{\epsilon_{b1}}{\epsilon_{b1}} &lt; \frac{1}{k^2 \rho_a} )</td>
</tr>
<tr>
<td>Panel (c): Complete</td>
<td>S</td>
<td>( \frac{\rho_{a}}{\rho_{a}} )</td>
<td>( \frac{\rho_{a}}{\rho_{a}} )</td>
</tr>
<tr>
<td>domestic  financial market  (Economies I, II, III)</td>
<td>N</td>
<td>( \frac{\rho_{a}}{\rho_{a}} )</td>
<td>( \frac{\rho_{a}}{\rho_{a}} )</td>
</tr>
<tr>
<td>B</td>
<td>( \frac{\rho_{a}}{\rho_{a}} )</td>
<td>( \frac{\rho_{a}}{\rho_{a}} )</td>
<td>( \frac{\rho_{a}}{\rho_{a}} )</td>
</tr>
</tbody>
</table>
Table II: Agents 2 and 3’s optimal consumption policies

<table>
<thead>
<tr>
<th>Panel (a): No domestic financial market (Economy I)</th>
<th>$\epsilon_{ai}/\epsilon_{bi}$</th>
<th>$c_{ai} - c_{ai}$</th>
<th>$c_{bi} - c_{bi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>$&gt; \frac{k}{k^2} \frac{p_i}{p_a}$</td>
<td>$\frac{1}{k} \frac{p_i}{p_a} \epsilon_{ai} - \frac{\epsilon_{ai}}{2} &lt; 0$</td>
<td>$k^2 \frac{p_a}{p_i} \epsilon_{ai} - \frac{\epsilon_{ai}}{2} &gt; 0$</td>
</tr>
<tr>
<td>N</td>
<td>$\in \left[ \frac{k^2}{k^2} \frac{p_i}{p_a}, \frac{1}{k^2} \frac{p_i}{p_a} \right]$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>$&lt; k^2 \frac{p_i}{p_a}$</td>
<td>$k^2 \frac{p_i}{p_a} \epsilon_{ai} - \frac{\epsilon_{ai}}{2} &gt; 0$</td>
<td>$\frac{1}{k} \frac{p_a}{p_i} \epsilon_{ai} - \frac{\epsilon_{ai}}{2} &lt; 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel (b): Incomplete domestic financial market (Economy II)</th>
<th>$\epsilon_{ai}/\epsilon_{bi}$</th>
<th>$c_{ai} - c_{ai}$</th>
<th>$c_{bi} - c_{bi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>$&gt; \frac{p_i}{k^2} \frac{p_a^<em>}{p_{a^</em>}}$</td>
<td>$\frac{1}{k} \frac{p_i}{p_{a^*}} \epsilon_{ai} - \frac{\epsilon_{ai}}{2} &lt; 0$</td>
<td>$k \frac{p_{a^*}}{p_i} \epsilon_{ai} - \frac{\epsilon_{ai}}{2} &gt; 0$</td>
</tr>
<tr>
<td>N</td>
<td>$\in \left[ \frac{1}{k} \frac{p_i}{p_{a^<em>}}, \frac{1}{k} \frac{p_i}{p_{a^</em>}} \right]$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>$&lt; \frac{p_i}{k^2} \frac{p_a^<em>}{p_{a^</em>}}$</td>
<td>$\frac{1}{k} \frac{p_i}{p_{a^*}} \epsilon_{ai} - \frac{\epsilon_{ai}}{2} &gt; 0$</td>
<td>$\frac{1}{k} \frac{p_{a^*}}{p_i} \epsilon_{ai} - \frac{\epsilon_{ai}}{2} &lt; 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel (c): Complete domestic financial market (Economies I*, III)</th>
<th>$\epsilon_{ai}/\epsilon_{bi}$</th>
<th>$c_{ai} - c_{ai}$</th>
<th>$c_{bi} - c_{bi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>$&gt; \frac{p_i}{k^2} \frac{p_{a^<em>}}{p_{a^</em>}}$</td>
<td>$\frac{p_{a^*}}{p_i} \epsilon_{ai} - \frac{\epsilon_{ai}}{2} &lt; 0$</td>
<td>$\frac{p_{a^*}}{p_i} \epsilon_{ai} - \frac{\epsilon_{ai}}{2} &gt; 0$</td>
</tr>
<tr>
<td>N</td>
<td>$= \frac{p_{a^<em>}}{p_{a^</em>}}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>$&lt; \frac{p_i}{k^2} \frac{p_{a^<em>}}{p_{a^</em>}}$</td>
<td>$\frac{p_{a^*}}{p_i} \epsilon_{ai} - \frac{\epsilon_{ai}}{2} &gt; 0$</td>
<td>$\frac{p_{a^*}}{p_i} \epsilon_{ai} - \frac{\epsilon_{ai}}{2} &lt; 0$</td>
</tr>
</tbody>
</table>

The second part of the solution (the sharing between any redundant securities) depends on the real exchange rate in the relevant state ($p_a/p_{a^*}$ or $p_b/p_{b^*}$). Given a consumption choice for state $a$, agent 1 is indifferent between all holdings such that:

$$\alpha_{a1} + \frac{p_a}{p_{a^*}} \alpha_{a^1} = c_{a1} - \epsilon_{a1}, \quad \text{where:} \quad \alpha_{a^1} = \begin{cases} 0 & \text{if } \frac{p_a}{p_{a^*}} = k; \\ \leq 0 & \text{if } \frac{p_a}{p_{a^*}} \in \left( 0, \frac{1}{k} \right); \\ \geq 0 & \text{if } \frac{p_a}{p_{a^*}} = \frac{1}{k}, \end{cases}$$

and analogously for state $b$. Agent $i \in \{2, 3\}$ is indifferent between all holdings such that:

$$\alpha_{ai} + \frac{p_{a^*}}{p_a} \alpha_{ai} = c_{ai} - \epsilon_{ai}, \quad \text{where:} \quad \alpha_{ai} = \begin{cases} 0 & \text{if } \frac{p_{a^*}}{p_a} = k; \\ \leq 0 & \text{if } \frac{p_{a^*}}{p_a} \in \left( 0, \frac{1}{k} \right); \\ \geq 0 & \text{if } \frac{p_{a^*}}{p_a} = \frac{1}{k}, \end{cases}$$

and analogously for state $b$.

**Equilibrium under Intranational Homogeneity.** An equilibrium is a set of prices for the available securities ($p_a, p_b$ in Economy I, etc.) and portfolio strategies ($\alpha_1, \alpha_2, \alpha_3$) such that $\alpha_i$ solves agent $i$’s problem, defined in (A.1)-(A.2), and all markets clear, i.e., $\sum_i \alpha_{ai} = \sum_i \alpha_{bi} = \sum_i \alpha_{a^1} = \sum_i \alpha_{b^1} = 0$.

Under intranational homogeneity (Section 2.2), all Economies I-III yield the same equilibrium allocations, as provided in Table III.
Table III: Conditions for the regions, relative prices, and equilibrium allocations

<table>
<thead>
<tr>
<th>Country 1</th>
<th>Country 2</th>
<th>Condition</th>
<th>( p_a^* / p_a )</th>
<th>( e_{a1} - e_{a2} )</th>
<th>( e_{a1} - e_{a2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>S</td>
<td>( e_{a1} / e_{b1} ) &lt; ( k^2 )</td>
<td>( k )</td>
<td>( k e_{a2} e_{b1} - e_{a1} e_{b2} / k )</td>
<td>&gt; 0</td>
</tr>
<tr>
<td>N</td>
<td>N</td>
<td>( e_{a1} / e_{b1} ) ( e_{a2} / e_{b2} ) ( \in \left[ k^2, \frac{1}{k^2} \right] )</td>
<td>( e_{a1} e_{b2} / k - k e_{a2} e_{b1} / 2(k e_{b1} + e_{b2}) )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>S</td>
<td>B</td>
<td>( e_{a1} / e_{b1} ) ( e_{a2} / e_{b2} ) &gt; ( k^2 )</td>
<td>( \frac{1}{k} )</td>
<td>( e_{a2} e_{b1} / k - k e_{a1} e_{b2} / 2(k e_{b1} + e_{b2}) )</td>
<td>&lt; 0</td>
</tr>
</tbody>
</table>

Equilibrium under Intranational Heterogeneity. We will discuss only Economies I, II and III since the removal of financial markets from the homogeneous country (Economy I*) has no effect. Henceforth, we denote each possible case by the situations of agents 2 and 3 (e.g., SB denotes agent 2 selling consumption and agent 3 buying consumption in state a); agent 1’s decision need not be considered since he faces a linear problem (and so the expressions for his demands are identical whatever his shipping situation).

Table IV reports the conditions for the occurrence of four of the nine possible equilibrium regions (SS, NN, BB, SB, BS, SN, NS, BN, NB); the remaining five regions are “mirror images”, obtained by swapping either agents 2 and 3 or states a and b.

Table IV: Conditions for the possible regions

<table>
<thead>
<tr>
<th>Cases</th>
<th>Economy I</th>
<th>Economy II</th>
<th>Economy III</th>
</tr>
</thead>
</table>
| SS    | \( \left\{ \begin{array}{l}
\frac{e_{a1}}{e_{b1}} > \frac{k e_{a2} + e_{a3}}{k e_{b2} + k e_{b3}} \\
\frac{e_{a2}}{e_{b2}} > \frac{k e_{a1} + e_{a3}}{k e_{b1} + k e_{b3}}
\end{array} \right. \) | \( \left\{ \begin{array}{l}
\frac{e_{a1}}{e_{b1}} > \frac{k e_{a2} + e_{a3}}{k e_{b2} + k e_{b3}} \\
\frac{e_{a2}}{e_{b2}} > \frac{k e_{a1} + e_{a3}}{k e_{b1} + k e_{b3}}
\end{array} \right. \) | \( \left\{ \begin{array}{l}
\frac{e_{a1}}{e_{b1}} > \frac{k e_{a2} + e_{a3}}{k e_{b2} + k e_{b3}} \\
\frac{e_{a2}}{e_{b2}} > \frac{k e_{a1} + e_{a3}}{k e_{b1} + k e_{b3}}
\end{array} \right. \) |
| SB    | \( \left\{ \begin{array}{l}
\frac{e_{a1}}{e_{b1}} > \frac{e_{a2} / k^2 + e_{a3}}{k e_{b2} + k e_{b3}} \\
\frac{e_{a2}}{e_{b2}} > \frac{e_{a1} / k^2 + e_{a3}}{k e_{b1} + k e_{b3}}
\end{array} \right. \) | \( \left\{ \begin{array}{l}
\frac{e_{a1}}{e_{b1}} > \frac{e_{a2} / k^2 + e_{a3}}{k e_{b2} + k e_{b3}} \\
\frac{e_{a2}}{e_{b2}} > \frac{e_{a1} / k^2 + e_{a3}}{k e_{b1} + k e_{b3}}
\end{array} \right. \) | \( \left\{ \begin{array}{l}
\frac{e_{a1}}{e_{b1}} > \frac{e_{a2} / k^2 + e_{a3}}{k e_{b2} + k e_{b3}} \\
\frac{e_{a2}}{e_{b2}} > \frac{e_{a1} / k^2 + e_{a3}}{k e_{b1} + k e_{b3}}
\end{array} \right. \) |
| SN    | \( \left\{ \begin{array}{l}
\frac{e_{a1}}{e_{b1}} > \frac{e_{a2} / k^2 + e_{a3}}{k e_{b2} + k e_{b3}} \\
\frac{e_{a2}}{e_{b2}} > \frac{e_{a1} / k^2 + e_{a3}}{k e_{b1} + k e_{b3}}
\end{array} \right. \) | \( \left\{ \begin{array}{l}
\frac{e_{a1}}{e_{b1}} > \frac{e_{a2} / k^2 + e_{a3}}{k e_{b2} + k e_{b3}} \\
\frac{e_{a2}}{e_{b2}} > \frac{e_{a1} / k^2 + e_{a3}}{k e_{b1} + k e_{b3}}
\end{array} \right. \) | \( \left\{ \begin{array}{l}
\frac{e_{a1}}{e_{b1}} > \frac{e_{a2} / k^2 + e_{a3}}{k e_{b2} + k e_{b3}} \\
\frac{e_{a2}}{e_{b2}} > \frac{e_{a1} / k^2 + e_{a3}}{k e_{b1} + k e_{b3}}
\end{array} \right. \) |
| NN    | \( \left\{ \begin{array}{l}
\frac{k^2 e_{a1}}{e_{b1}} > \frac{e_{a1}}{e_{b1}} > \frac{e_{a3}}{k e_{b3}} \\
\frac{k^2 e_{a2}}{e_{b2}} > \frac{e_{a2}}{e_{b2}} > \frac{e_{a3}}{k e_{b3}}
\end{array} \right. \) | \( \left\{ \begin{array}{l}
\frac{k^2 e_{a1}}{e_{b1}} > \frac{e_{a1}}{e_{b1}} > \frac{e_{a3}}{k e_{b3}} \\
\frac{k^2 e_{a2}}{e_{b2}} > \frac{e_{a2}}{e_{b2}} > \frac{e_{a3}}{k e_{b3}}
\end{array} \right. \) | \( \left\{ \begin{array}{l}
\frac{k^2 e_{a1}}{e_{b1}} > \frac{e_{a1}}{e_{b1}} > \frac{e_{a3}}{k e_{b3}} \\
\frac{k^2 e_{a2}}{e_{b2}} > \frac{e_{a2}}{e_{b2}} > \frac{e_{a3}}{k e_{b3}}
\end{array} \right. \) |
This table verifies that the conditions for occurrence of the various possible cases indeed are affected by the introduction of the new securities \( a^* \) and \( b^* \), thus exhibiting a role for financial innovation in the circumvention of imperfections in the good market. The evolution in occurrence (in terms of the distribution of endowments) of the regions is summarized in Table V.

**Table V: Evolution across economies of the size of the possible regions**

<table>
<thead>
<tr>
<th>Situations of agents 2 &amp; 3</th>
<th>Eco. I → Eco. II</th>
<th>Eco. II → Eco. III</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS/BB</td>
<td>remains the same</td>
<td>remains the same</td>
</tr>
<tr>
<td>SB/BS</td>
<td>grows larger</td>
<td>grows larger</td>
</tr>
<tr>
<td>SN/NS/BN/NB</td>
<td>shrinks</td>
<td>becomes “knife-edge”</td>
</tr>
<tr>
<td>NN</td>
<td>remains the same</td>
<td>becomes “knife-edge”</td>
</tr>
</tbody>
</table>

Table V confirms that financial innovation within country 2 facilitates risk-sharing between agents 2 and 3; no-trade regions shrink and trade regions grow.

**Appendix B: Proofs**

**Proof of Proposition 4.1:** From the definition of \( e_i \), agent \( i \)'s consumption is given by \( f(e_i(t), t) \equiv e_i(t) + (e_i(t))^+/k - k(e_i(t))^− \). Agent \( i \)'s dynamic budget constraint (4.1) implies (together with \( X_i(T) \geq 0 \)) that \( e_i \) satisfies the static budget constraint \( E \left[ \int_0^T \xi(t)e_i(t)dt \right] \leq 0 \). Hence agent \( i \) effectively solves the problem

\[
\max_{e_i} E \left[ \int_0^T u_i (f(e_i(t), t)) \, dt \right] \quad \text{subject to} \quad E \left[ \int_0^T \xi(t)e_i(t)dt \right] \leq 0,
\]

whose first-order conditions are (4.2)-(4.3). *Q.E.D.*

**Proof of Proposition 4.2:** For any given \( \Phi_i(t) \), agent \( i \) chooses the security holdings that maximize the value of the drift term in (4.5) or (4.7) (if he did not do so, he could increase his consumption at no extra cost or additional risk-taking). A necessary condition for this maximization problem to have a solution is (4.9). If it holds, substitution into (4.5) or (4.7) reveals that all holdings verifying (4.8) lead to the same dynamics for \( X_i \). *Q.E.D.*

**Proof of Proposition 4.3:** Agent 1 faces a complete, perfect market problem and so the solution to his problem obtains using standard techniques (Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987)). For \( i \in \{2, 3\} \), agent \( i \)'s consumption “expenditures” (in units of country 1 good) can be defined as \( e_i(t) \equiv p(t)(c_i(t) - e_i(t)) \), in the sense that this is the cash-flow he needs to replicate to finance the consumption plan \( c_i \). He effectively solves the problem

\[
\max_{e_i} E \left[ \int_0^T u_i \left( e_i(t) + \frac{e_i(t)}{p(t)} \right) \, dt \right] \quad \text{subject to} \quad E \left[ \int_0^T \xi(t)e_i(t)dt \right] \leq 0,
\]

whose first-order conditions are (4.11)-(4.12). *Q.E.D.*
Proof of Proposition 5.1: Assume that equilibrium exists. Clearing in the consumption good implies (5.6), while (5.5) follows from agent 1’s optimality and the definition of \( p \), and (5.4) obtains by substitution of the expression for \( \xi \) given in (5.5) and algebraic manipulation of agent 1’s static budget constraint. It is rational for country 1 to import only if \( p(t) = k \), hence (5.1). The existence of \( x^I(t) > 0 \) satisfying (5.1) requires \( y_1u_2'(\epsilon_2(t)) / y_2u_1'(\epsilon_1(t)) < k \), hence the condition for country 1 importing. The case of country 1 exporting is symmetric. Whenever none of the conditions for country 1 importing or exporting holds (i.e., \( k \leq y_1u_2'(\epsilon_2(t)) / y_2u_1'(\epsilon_1(t)) \leq 1/k \)), there does not exist any amount of international trade \( x^I(t) > 0 \) or \( x^E(t) > 0 \) that is rational, and so countries are in autarky. The individual-specific state price densities must be such that each agent is content to consume his own endowment, hence (5.2).

Conversely, assume that (5.1)-(5.5) hold. From (5.4) and the first equality in (5.5), country 1 finds it optimal to consume as in the first equality of (5.6), while from the second equality of (5.5), and the values of \( p(t) \) in (5.1), (5.2) and (5.3) country 2 optimally consumes as in the second equality of (5.6). Hence, the good market clears (given that, additionally, the conditions for the three possible cases cover the whole space of possible values for \( u_2'(\epsilon_2(t)) / u_1'(\epsilon_1(t)) \)).

It remains to verify that the corresponding portfolio holdings clear the financial markets. Take the case of Economy I. At any time \( t \), standard arguments imply that agents’ wealths verify

\[
X_1(t) = \frac{1}{\xi(t)} E \left[ \int_t^T \xi(s)(c_1(s) - \epsilon_1(s))ds | {\mathcal F}_t \right] = \frac{1}{\xi(t)} E \left[ \int_t^T \xi(s) (x^I(s) - x^E(s)) ds | {\mathcal F}_t \right],
\]

\[
X_2(t) = \frac{1}{\xi(t)} E \left[ \int_t^T \xi(s)(e_2(s))ds | {\mathcal F}_t \right] = \frac{1}{\xi(t)} E \left[ \int_t^T \xi(s) \left\{ \frac{1}{k}(e_2(s) - c_2(s))^+ - k(e_2(s) - c_2(s))^- \right\} ds | {\mathcal F}_t \right].
\]

Hence, clearing in the good market implies that \( X_1(t) + X_2(t) = 0 \). On the other hand, using standard arguments and the martingale representation theorem, agents’ holdings in the risky security \( P \) are given by:

\[
\pi_{P_i} = \frac{\mu(t) - r(t)}{\sigma(t)^2} X_i(t) + \frac{\kappa_i(t)}{\sigma(t)\xi(t)},
\]

where \( \kappa_1, \kappa_2 \) satisfy

\[
\xi(t)X_1(t) + \int_0^t \xi(s)(c_1(s) - \epsilon_1(s))ds = \int_0^t \kappa_1(s)dW(s),
\]

\[
\xi(t)X_2(t) + \int_0^t \xi(s)c_2(s)ds = \int_0^t \kappa_2(s)dW(s).
\]

Clearing in the good market thus implies that, \( \forall t, \int_0^t (\kappa_1(s) + \kappa_2(s))dW(s) = 0 \), so that \( \kappa_1(t) + \kappa_2(t) = 0, \forall t, \) and \( \pi_{P_1(t)} + \pi_{P_2(t)} = 0 \): clearing in the “stock” \( (P) \) market. \( X_1(t) + X_2(t) = 0 \) then implies clearing in the “bond” \( (M) \). The case of Economy I* is symmetric. In Economies II and III, due to the presence of redundant securities there exist a continuum of portfolio strategies consistent with agents’ consumption choices, among which the agents are indifferent by Proposition 4.2. In particular, the agents are satisfied by holdings in \( M \) and \( P \) as in Economy I above (and zero holdings in \( M^* \) and \( P^* \)), so there exist security holdings that allow them to implement their consumption choices and clear markets. Q.E.D.
Proof of Corollary 5.1: For all regions, for any \( t \), the equilibrium allocations of Proposition 5.1 can readily be checked to coincide with the solution of (5.7). \textit{Q.E.D.}

Proof of Proposition 5.2: Applying the implicit function theorem to (5.1) yields the derivatives of \( x^t \) with respect to \( \epsilon_1 \) and \( \epsilon_2 \). Applying Itô’s lemma then yields, after substitution of (5.9), the expressions in the Proposition. \textit{Q.E.D.}

Proof of Proposition 5.3: Agents’ consumption dynamics follows from Proposition 5.2. Applying Itô’s lemma to both sides of their first-order condition and identifying drifts and diffusions then yields, after substitution of (5.9) and (5.10), (5.11)-(5.14). \textit{Q.E.D.}

Proof of Corollary 5.2: (5.15)-(5.16) follow from applying Itô’s lemma to the right hand side of \( p(t) = y_1 u'_2(\epsilon_2(t)) / y_2 u'_1(\epsilon_1(t)) \) and identifying drifts and diffusions. \textit{Q.E.D.}

Proofs of Propositions 6.1-6.2: These are extensions to the case of three agents of the proofs of Proposition 5.1-5.3 above. \textit{Q.E.D.}

Proof of Proposition 6.3: If an equilibrium exists, from Section 4.2, \( c_2 \) and \( c_3 \) are such that \( u'_2(c_2(t))/y_2 = u'_3(c_3(t))/y_3 = (p(t)\xi(t)) \). Additionally using good market clearing (3.3), \( \forall t \), agents 2 and 3’s consumption allocations solve the problem

\[
\max_{c_2(t) + c_3(t) = \epsilon_2(t) + \epsilon_3(t) + k x^E(t) - x^I(t)/k} u_2(c_2(t)) + \frac{y_2}{y_3} u_3(c_3(t)),
\]

i.e., the allocations solve the representative agent problem (6.10). Substitution into Proposition 5.1 then yields the equilibrium conditions (6.9)-(6.14). (The additional budget constraint (6.12), necessary to pin down the allocation within country 2, is agent 2’s static budget constraint.)

Conversely, if (6.9)-(6.13) hold, from Proposition 5.1 there is an equilibrium involving agent 1 on the one hand, and the country-2 representative agent, with utility \( u^* \) defined in (6.10) and endowed with the process \( \epsilon_1 + \epsilon_2 \), on the other hand. All that remains to be checked is that agents 2 and 3 are content with the allocations from the representative agent problem (6.10). These are such that

\[
\frac{u'_2(c_2(t))}{y_2} = \frac{u'_3(c_3(t))}{y_3} = u^*(\epsilon_2(t) + \epsilon_3(t) + k x^E(t) - \frac{x^I(t)}{k}) = p(t)\xi(t),
\]

where the last equality follows from the country-2 representative agent optimality. This implies (together with satisfaction of the individual budget constraints (6.11)-(6.12)) that these consumption allocations are optimal for agents 2 and 3. \textit{Q.E.D.}
References


