Optimal Life Cycle Asset Allocation: Understanding the Empirical Evidence

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Abstract

We show that a life cycle model with realistically calibrated uninsurable labor income risk and moderate risk aversion can simultaneously match stock market participation rates and asset allocation decisions conditional on participation. The key ingredients of the model are Epstein-Zin preferences, a fixed stock market entry cost, and moderate heterogeneity in risk aversion. Households with low risk aversion smooth earnings shocks with a small buffer stock of assets and consequently most of them (optimally) never invest in equities. Therefore, the marginal stockholders are (endogenously) more risk averse and as a result they do not invest their portfolios fully in stocks.

JEL Classification: G11.

In this paper we present a life cycle asset allocation model with intermediate consumption and stochastic uninsurable labor income, that provides an explanation for two very important empirical observations: low stock market participation rates in the population as a whole, and moderate equity holdings for stock market participants.

Our life cycle model integrates three main motives that have been identified as quantitatively important in explaining individual and aggregate wealth accumulation. First, a precautionary savings motive in the presence of undiversifiable labor income risk generates asset accumulation to smooth unforeseen contingencies (Deaton (1991) and Carroll (1992, 1997)). Second, pension income is lower than mean working-life labor income implying that saving for retirement becomes important at some point in the life cycle. The combination of precautionary and retirement saving motives has recently been shown to generate realistic wealth accumulation profiles over the life cycle. First, we explicitly incorporate a bequest motive which has recently been shown to be important in matching the skewness of the wealth distribution (de Nardi (forthcoming) and Laitner (2002)).

More recently, life cycle models incorporating some (or all) of these motives have been extended to include an asset allocation decision, both in an infinite-horizon and in a finite-horizon, life cycle setting. However, several important predictions of these models are still at odds with empirical regularities. First, low stock market participation in the population (Mankiw and Zeldes (1991)) persists. The latest Survey of Consumer Finances (2001) reports that only 52 percent of US households hold stocks either directly or indirectly (through pension funds, for instance), while these models predict that, given the equity premium, all households should participate in the stock market as soon as saving takes place. Second,

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households in the model invest almost all of their wealth in stocks, in contrast to both casual empirical observation, and to formal empirical evidence (see Poterba and Samwick (1999) or Ameriks and Zeldes (2001), for instance).

We develop a life cycle asset allocation model that tries to address these two puzzles. We argue that it is possible to simultaneously match stock market participation rates and asset allocation conditional on participation, with moderate values of risk aversion (between one and five), and without extreme assumptions about the level of background risk. Our model has three key features. First, we include a fixed entry cost for households that want to invest in risky assets for the first time. A large literature has concluded that some level of fixed costs seems to be necessary to improve the empirical performance of asset pricing models.4 Since the excessive demand for equities predicted by asset allocation models is merely the portfolio-demand manifestation of the equity premium puzzle, introducing a fixed cost in the model seems to be a natural extension. Moreover, recent empirical work suggests that small entry costs can be consistent with the observed low stock market participation rates (see Paiella (2001), Degeorge, Jenter, Moel and Tufano (2002) and Vissing-Jørgensen (2002b)).

The other two key features are motivated by the (perhaps surprising) implication of the model that participation rates are an increasing function of risk aversion, at least over a wide range of parameter values. Specifically, changing risk aversion generates two opposing forces for determining the participation decision. On one hand, more risk averse households optimally prefer to invest a smaller fraction of their wealth in stocks. On the other hand, risk aversion determines prudence and more prudent consumers accumulate significantly more wealth over the life cycle. We show that the higher wealth accumulation motive dominates for moderate coefficients of relative risk aversion (that is, not greater than five). As a result, the less risk averse investors have a weaker incentive to pay the fixed cost. This explains why previous attempts to match participation rates in the context of a life cycle model were fairly unsuccessful. If we try to match asset allocation decisions by assuming high values of risk aversion, the implied participation rates are counterfactually high (e.g. Campbell et al. (2001)). Motivated by this result we allow for preference heterogeneity in the population,

the second key feature of the model. As argued before, since the less risk averse investors accumulate less wealth over the life cycle, the majority optimally chooses not to pay the fixed cost. Therefore, endogenously stock market participants tend to be the more risk averse investors and, consequently, even after paying the fixed cost they do not invest their portfolios fully in equities.

The final important feature of the model is the assumption of Epstein-Zin preferences, which allows us to separate risk aversion from the elasticity of intertemporal substitution (EIS). In the context of a life cycle model with labor income, wealth accumulation is a crucial determinant of both the stock market participation and the asset allocation decision. Within the power utility framework, households with low risk aversion also have a high EIS. Given that the expected return from investing in the stock market is higher than the discount rate, a higher EIS increases savings. As a result, even though the less risk averse agents would not save much for precautionary reasons, they would have a strong incentive to save for retirement (and for a potential bequest motive). Thus, breaking the link between risk aversion and the EIS is crucial for delivering predictions that are consistent with the observed empirical evidence.

Therefore, in our model, households with very low risk aversion and low (moderate) EIS smooth idiosyncratic earnings shocks with a small buffer stock of assets, and most of them never invest in equities (thus behaving as in the Deaton (1991) infinite horizon model). This seems to describe adequately the behavior of a large fraction of the U.S. population that retires without significant financial assets (and does not participate in the stock market). Within the low EIS and low risk aversion group, only a small fraction owns stocks, and they do so only as they get close to retirement. On the other hand, investors with high prudence and high EIS are the ones that participate in the stock market from early on, since they accumulate more wealth and therefore have a stronger incentive to pay the fixed cost. Therefore, the marginal stockholders are (endogenously) more risk averse and as a result they do not invest their portfolios fully in stocks.

\footnote{It is important to point out that we do not need heterogeneity in the EIS to obtain our results. As we will show, the less risk-averse investors can have the same EIS as the more risk-averse, just as long as this value is not too high (hence the need for Epstein-Zin preferences).}
The heterogeneous agent model can simultaneously match the stock market participation rate and the average equity allocation conditional on participation, from the *Survey of Consumer Finances*. The life cycle profile of the participation rate is also very close to the one observed in the data. On the negative side, the model still counterfactually predicts that young households which have already paid the participation cost will invest most of their portfolio in equities.\(^6\) Finally, the degree of heterogeneity in the wealth distribution is quite comparable to the one observed in the data.

The rest of the paper is organized as follows. Section I summarizes results from the existing empirical literature on life cycle asset allocation while section II outlines the model and calibration. In Section III (IV) we discuss the results in the absence (presence) of the fixed entry cost, and section V concludes.

## I Empirical Evidence on Life Cycle Asset Allocation and Stock Market Participation

In most industrialized countries, stock market participation rates have increased substantially during the last decade. Nevertheless, a large percentage of the population still does not own any stocks (either directly or indirectly through pension funds). Moreover, even those households which do own stocks, still invest a significant fraction of their portfolios in alternative assets.

Figures 1.1 and 1.2 summarize evidence reported in Ameriks and Zeldes (2001).\(^7\) The results are sensitive to the identifying assumptions regarding time versus cohort effects. Time effects can arise, for example, from changes in market structure (e.g. transaction costs or information) or because investors use past returns to forecast future expected returns. Cohort effects can be due to differences in life time earnings potential, or different institutional settings (e.g. the social security system). Since age \((a)\), time \((t)\) and cohort \((c)\), birth

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\(^6\)Hu (2001) and Yao and Zhang (2002) are able to reduce the equity demand of young households by considering models with an explicit housing allocation decision.

\(^7\)Guiso, Haliassos and Japelli (2002) obtain similar conclusions using cross-sectional information for five different countries (U.S.A., U.K., Netherlands, Germany and Italy).
year) are linearly dependent \((a \equiv t - c)\), when constructing age profiles it is impossible to simultaneously identify time and cohort effects.

Figure 1.1 plots the average life cycle equity holdings for stock market participants (as a share of total financial wealth), based on the 1989, 1992, 1995 and 1998 waves of the Survey of Consumer Finances (SCF). Although the life cycle profiles are very sensitive to the inclusion of time dummies versus the inclusion of cohort dummies, the average stock holdings are significantly below 100 percent in both cases.\(^8\) Figure 1.2 plots the corresponding stock market participation rate, obtained by running a Probit regression on the same data. These results are less sensitive to the choice of time versus cohort dummies. As expected, a very large fraction of the population does not own equities. In both cases the participation rate gradually increases until approximately age 50. When including cohort dummies, the profile is flat after age 50, while with time dummies it is decreasing. Ameriks and Zeldes (2001) obtain the same results after re-doing the analysis using TIAA-CREF data from 1987-1996, and so do Poterba and Samwick (1999), using SCF data.

We can summarize the existing evidence as follows.\(^9\) First, the stock market participation rate in the U.S. population is close to 50 percent. Using the latest numbers from the SCF we compute it as 51.9 percent (details given in Appendix C). Second, participation rates increase during working life and there is some evidence suggesting that they might decrease during retirement although this might also be due to cohort effects. Third, conditional on stock market participation, households invest a large fraction of their financial wealth in alternative assets. According to the latest numbers from the SCF, the average equity holdings as a share of financial wealth for stock market participants, is 54.8 percent. Fourth,

\(^8\)The OLS regression with cohort effects predicts a share of financial wealth invested in stocks above 100% for the oldest age groups. This is just the result of imposing the same cohort effects on the full sample as in fact, in every individual cross-section, these age groups never invests more than 60% of their wealth in equities.

\(^9\)We must point out that several papers have contributed to this research. See for example, Guiso, Jappelli and Terlizzese (1996) (who focus mostly on the impact of background risk on asset allocation), King and Leape (1998), Heaton and Lucas (2000) and the papers in the volume edited by Guiso, Haliassos and Japelli (2002).
there is no clear pattern of equity holdings over the life cycle.

II The Model

A Preferences

Time is discrete and \( t \) denotes adult age which, following the typical convention in this literature, corresponds to effective age minus 19. Each period corresponds to one year and agents live for a maximum of 81 (\( T \)) periods (age 100). The probability that a consumer/investor is alive at time \( (t + 1) \) conditional on being alive at time \( t \) is denoted by \( p_t \) (\( p_0 \) equal to 1).

Households have Epstein-Zin utility functions (Epstein and Zin (1989)) defined over one single non-durable consumption good. Let \( C_t \) and \( X_t \) denote respectively consumption level and wealth (cash on hand) at time \( t \) then, the household’s preferences are defined by

\[
V_t = \{(1 - \beta p_t)C_t^{1-\psi} + \beta E_t \left[ p_t[V_{t+1}^{1-\gamma}] + (1 - p_t)b \frac{(X_{t+1}/b)^{1-\gamma}}{1 - \gamma} \right]^{\frac{1}{1-\psi}} \}^{\frac{1}{1-\gamma}} \tag{1}
\]

where \( \rho \) is the coefficient of relative risk aversion, \( \psi \) is the elasticity of intertemporal substitution, \( \beta \) is the discount factor, and \( b \) determines the strength of the bequest motive.\(^{10}\)

Given the presence of a bequest motive, the terminal condition for the recursive equation (1) is:

\[
V_{T+1} \equiv b \frac{(X_{T+1}/b)^{1-\rho}}{1 - \rho} \tag{2}
\]

B Labor Income Process

Following the standard specification in the literature, the labor income process before retirement is given by

\[
Y_{it} = P_{it}U_{it} \tag{3}
\]

\[
P_{it} = \exp(f(t, Z_{it}))P_{it-1}N_{it} \tag{4}
\]

\(^{10}\)For more motivation and details on the modelling of bequest motives in life-cycle models see Laitner (2002), or De Nardi (forthcoming).
where \( f(t, Z_{it}) \) is a deterministic function of age and household characteristics \( Z_{it} \), \( P_{it} \) is a permanent component with innovation \( N_{it} \), and \( U_{it} \) a transitory component. We assume that \( \ln U_{it} \) and \( \ln N_{it} \) are independent and identically distributed with mean \( \{-0.5*\sigma_u^2, -0.5*\sigma_n^2\} \), and variances \( \sigma_u^2 \) and \( \sigma_n^2 \), respectively. The log of \( P_{it} \), evolves as a random walk with a deterministic drift, \( f(t, Z_{it}) \).

For simplicity, retirement is assumed to be exogenous and deterministic, with all households retiring in time period \( K \), corresponding to age 65 (\( K = 46 \)). Earnings in retirement (\( t > K \)) are given by \( Y_{it} = \lambda P_{iK} \), where \( \lambda \) is the replacement ratio (a scalar between zero and one). This specification, also standard in this literature, considerably facilitates the solution of the model, as it does not require the introduction of an additional state variable (see section II.E).

Durable goods, and in particular housing, can provide an incentive for higher spending early in life. Modelling these decisions directly is beyond the scope of the paper, but nevertheless we take into account these potential patterns in life cycle expenditures. Using the P.S.I.D., for each age \( (t) \) we estimate the percentage of household income that is dedicated to housing expenditures \( (h_t) \) and subtract it from the measure of disposable income.\(^\text{11}\) More details on this estimation are given below, when we discuss the calibration of the model.

C Financial Assets

The investment opportunity set is constant and there are two financial assets, one riskless (treasury bills or cash) and one risky (stocks). The riskless asset yields a constant gross return, \( R^f \), while the return on the risky asset (denoted by \( R^S_t \)) is given by

\[
R^S_{t+1} - R^f = \mu + \varepsilon_{t+1}
\]

where \( \varepsilon_t \sim N(0, \sigma^2) \).

We allow for positive correlation between stock returns and earnings shocks. We let \( \phi_N \) (\( \phi_U \)) denote the correlation coefficient between stock returns and permanent (transitory) income shocks.

\(^{11} \)A similar approach is taken by Flavin and Yamashita (2002) in a model without labor income.
Before investing in stocks for the first time, the investor must pay a fixed lump sum cost, \( F \times P_{it} \). This entry fee represents both the explicit transaction cost from opening a brokerage account and the (opportunity) cost of acquiring information about the stock market. The fixed cost \( F \) is scaled by the level of the permanent component of labor income \( P_{it} \) as this simplifies significantly the solution of the model. However, this specification is also motivated by the interpretation of the entry fee as the opportunity cost of time.

D Wealth accumulation

Following Deaton (1991) we denote cash on hand as the liquid resources available for consumption and saving. We define a dummy variable \( I_P \) which is equal to one when the fixed entry cost is incurred for the first time and zero otherwise. The household’s next period cash on hand \( X_{i,t+1} \) is given by

\[
X_{i,t+1} = S_{it}R^S_{t+1} + B_{it}R^f_{t+1} + (1 - h_t)Y_{i,t+1} - FIP_{i,t+1}
\]  

(6)

where \( S_{it} \) and \( B_{it} \) denote respectively stock holdings, and riskless asset holdings (cash) at time \( t \), and \( h_t \) is the fraction of income dedicated to housing related expenditures. Since the household must allocate cash on hand \( (X_{it}) \) between consumption expenditures \( (C_{it}) \) and savings we also have

\[
X_{it} = C_{it} + S_{it} + B_{it}
\]  

(7)

Finally, as in Deaton (1991), we prevent households from borrowing against their future labor income. More specifically we impose the following restrictions:

\[
B_{it} \geq 0
\]  

(8)

\[
S_{it} \geq 0
\]  

(9)

E The optimization problem and solution method

The complete optimization problem is then

\[
MAX_{(S_{it}, B_{it})_{t=1}^T} E(V_0)
\]  

(10)
where \( V_0 \) is given by equations (1) and (2); subject to the constraints given by equations (5) to (9), and to the stochastic labor income process given by (3) and (4) if \( t \leq K \), and 
\[ Y_{it} = \lambda P_{IK} \text{ if } t > K. \]

Analytical solutions to this problem do not exist. We therefore use a numerical solution method based on the maximization of the value function to derive the optimal decision rules. The details are given in appendix A, and here we just present the main idea. We first simplify the solution by exploiting the scale-independence of the maximization problem and rewriting all variables as ratios to the permanent component of labor income (\( P_{it} \)). The laws of motion and the value function can then be rewritten in terms of the normalized variables, and we use lower case letters to denote them (for instance, \( x_{it} \equiv \frac{X_{it}}{P_{it}} \)). This allows us to reduce the number of state variables to three: age (\( t \)), normalized cash on hand (\( x_{it} \)) and participation status (whether the fixed cost has already been paid or not). In the last period the policy functions are determined by the bequest motive and the value function corresponds to the bequest function. We can now use this value function to compute the policy rules for the previous period and, given these, obtain the corresponding value function. This procedure is then iterated backwards.

### F Computing Transition Distributions

After solving for the optimal policy functions, we can simulate the model to replicate the behavior of a large number of households and compute, for example, the corresponding average allocations. Here we propose an alternative method of computing various statistics that is based on the explicit calculation of the transition distribution of cash on hand from one age to the next. The computational details are delegated to Appendix B, but the intuitive idea is straightforward. Once we have solved for the policy functions we can substitute those in the budget constraint to obtain the distribution of \( x_{t+1} \) as a function of \( x_t \). Doing this for every possible \( x_t \) we are effectively computing the full transition matrix.\(^{12}\)

\(^{12}\) The results in the paper were computed both from the transition distributions and using Monte-Carlo simulations. The results were found to be identical, as long as the number of simulations is not too small (2000 or more).
Once we have these distributions, the unconditional mean consumption for age $t$ can then be computed as$^{13}$

$$\bar{c}_t = \theta_t \left\{ \sum_{j=1}^{J} \pi^I_{t,j} * c^I(x_j, t) \right\} + (1 - \theta_t) \left\{ \sum_{j=1}^{J} \pi^O_{t,j} * c^O(x_j, t) \right\}$$  \hspace{1cm} (11)$$

where $J$ is the number of grid points used in the discretization of normalized cash on hand, and $\pi^I_{t,j}$ and $\pi^O_{t,j}$ are the probability masses associated with each grid point at time $t$, for stockholders and non-stockholders, respectively. The participation rate at age $t$ ($\theta_t$) is given by

$$\theta_t = \theta_{t-1} + (1 - \theta_{t-1}) * \sum_{x_j > x^*} \pi^O_{t,j}$$  \hspace{1cm} (12)$$

where $x^*$ is the trigger point that causes participation, which is determined endogenously through the participation decision rule.

Finally, if we use $\alpha_t$ to denote the share of liquid wealth invested in the stock market at age $t$, then the unconditional portfolio allocation is computed as:

$$\bar{\alpha}_t = \frac{\theta_t * \left\{ \sum_{j=1}^{J} \pi^I_{t,j} * \alpha(x_j, t) * (x_j - c^I(x_j, t)) \right\}}{\theta_t * \sum_{j=1}^{J} \pi^I_{t,j} * (x_j - c^I(x_j, t)) + (1 - \theta_t) * \sum_{j=1}^{J} \pi^O_{t,j} * (x_j - c^O(x_j, t))}$$  \hspace{1cm} (13)$$

G Parameter Calibration

G.1 Preference parameters

We will start by presenting results for a relatively standard choice, (risk aversion) $\rho$ equal to five, (EIS) $\psi = 0.2$ and (discount factor) $\beta = 0.96$. However, later on we report results for several different values of both the coefficient of relative risk aversion ($\rho$) and the elasticity of intertemporal substitution ($\psi$), as these parameters will have very important implications for our results. We use the mortality tables of the National Center for Health Statistics to parameterize the conditional survival probabilities.

The importance of the bequest motive ($b$) is set at 2.5. As we discuss below, this parameter choice is motivated by the desire to match the wealth accumulation profiles observed in the data, but we will present some sensitivity analysis with respect to this parameter.

$^{13}$Superscript $I$ denotes households participating in the stock market, while superscript $O$ denotes households out of the stock market.
G.2 Labor income process

The deterministic labor income profile \( f(t, Z_{it}) \) reflects the hump shape of earnings over the life cycle, and the corresponding parameter values, just like the retirement transfers (\( \lambda \)), are taken from Cocco, Gomes and Maenhout (1999). With respect to standard deviations of the idiosyncratic shocks, the estimates range from 0.35 for \( \sigma_u \) and 0.12 for \( \sigma_n \) (Cocco et al. (1999)) to 0.1 for \( \sigma_u \) and 0.08 for \( \sigma_n \) (Carroll (1992)). We use numbers similar to the ones in Gourinchas and Parker (2002): \( \sigma_u = 0.15 \) and \( \sigma_n = 0.1 \). It is common practice to estimate different labor income profiles for different education groups (college graduates, high-school graduates, households without a high-school degree). In our paper we only report the results obtained with the parameters estimated from the sub-sample of high-school graduates, as the results for the other two groups are very similar.

G.3 Asset returns, correlation and fixed cost

The constant net real interest rate \( (R_f - 1) \) is set at two percent, while for the stock return process we consider a mean equity premium (\( \mu \)) equal to four percent and a standard deviation (\( \sigma_\varepsilon \)) of 18 percent. Considering an equity premium of four percent (as opposed to the historical six percent) is a fairly common choice in this literature (e.g. Yao and Zhang (forthcoming), Cocco (2001) or Campbell et al. (2001)). Even after having paid the fixed entry cost, the average retail investor still faces non-trivial transaction costs, mostly in the form of mutual fund fees. This adjustment is a short-cut representation for those costs, since the dimensionality of the problem prevents us from modelling them explicitly (as in Heaton and Lucas (1996), for example).

The evidence on the magnitude of the correlation between stock returns and permanent labor income shocks is mixed.\(^{14}\) Davis and Willen (2001) and Heaton and Lucas (2000) do not distinguish between the two components of labor income (permanent and transitory) when computing the correlation coefficients. For the purposes of calibrating our model we

\(^{14}\)Moreover, it has been argued that these estimations suffer from a small sample bias since the time-series dimension is too short in micro-data, and estimations using macro data usually yield larger and more significant correlations (see, for example, Jermann (1999)).
need to know the magnitude of the correlation coefficient for these two shocks separately. Campbell et al. (2001) estimate the correlation between the permanent component of labor income shocks and stock returns, and obtain a correlation coefficient of 0.15. They do not estimate a correlation between transitory shocks and stock returns and just assume it to be equal to zero. We will use these numbers ($\phi_N = 0.15$ and $\phi_U = 0.0$) for our benchmark calibration, and perform sensitivity analysis around these values.

With respect to the fixed cost of participation we will consider two limit cases: one where the cost is zero, and one where it equals 0.025 (2.5 percent of the household’s expected annual income). This parameter reflects both the monetary cost associated with the initial investment in the stock market, and the opportunity cost associated with obtaining the necessary information for making such investment.

G.4 Housing expenditures

We measure housing expenditures using data from the Panel Study of Income Dynamics from 1976 until 1993. For each household, in each year, we compute the ratio of annual mortgage payments and rent payments (housing related expenditures - $H$) relative to annual labor income ($Y$):

$$h_{it} \equiv \frac{H_{it}}{Y_{it}}$$  \hspace{1cm} (14)

\hspace{1cm}

15 It is important to realize that, in their tables Campbell et al. (2001) actually report the correlation of the aggregate component of permanent labor income shocks with stock returns. This explains their high estimates: 45.6%. To obtain the correlation with the “total permanent shock”, we need to adjust for the standard deviation of the aggregate component relative to the total, which gives the 15% number.

16 Consider the average household which has an annual labor income of $35000. If the time cost were zero, then this value of $F$ would imply a monetary cost of $875. If instead the monetary cost were zero, then this would imply a time cost of 9.1 days (6.3 working days). More generally, any convex combination of these two costs is acceptable. For example, a time cost of one (two) day(s) and a monetary cost of $779 ($683). Paiella (2001) and Vissing-Jorgensen (2002b) have used Euler equation estimation methods to obtain implied participation costs from observed consumption choices. They find values in the $75 to $200 range, but these are per-period costs, so our number is quite reasonable when compared to their estimates.

17 Before 1976 there is no information on mortgage expenditures, and 1993 is the last year available on final release from the PSID.
We combine mortgage payments and rent together since we are not modelling the housing decision explicitly. We identify the age effects by running the following regression on the full panel:

\[ h_{it} = A + B_1 \times \text{age} + B_2 \times \text{age}^2 + B_3 \times \text{age}^3 + \text{time dummies} + \zeta_{it} \]  

(15)

where age is defined as the age of the head of the household. We eliminate all observations with age greater than 75.\(^{18}\) The estimation results are reported in Table I.

In the model we use

\[ h_t = \max(A + B_1 \times \text{age} + B_2 \times \text{age}^2 + B_3 \times \text{age}^3, 0) \]  

(16)

which, given our parameter estimates, truncates \( h_t \) at zero for \( \text{age} \geq 80 \).

III Results without the Fixed Participation Cost

A Consumption and wealth accumulation

Figure 2.1 plots mean normalized consumption (\( \overline{c}_t \)), mean normalized wealth (\( \overline{w}_t \)) and mean normalized income net of housing expenditures ((\( 1 - h_t \)) \(*\overline{y}_t \)). The preference parameters are \( \rho \) equal to five and \( \psi \) equal to 0.2, and the importance of the bequest motive (\( b \)) is set at 2.5. Early in life the household is liquidity constrained and saves only a small buffer stock of wealth. From approximately ages 30 to 35 onwards, she starts saving for retirement and bequests, and wealth accumulation increases significantly. During retirement, consumption decreases as a result of the very high effective discount rate (high mortality risk). Wealth does not fall towards zero due to the presence of the bequest motive.\(^{19}\)

Table II shows the mean consumption to wealth ratio for different values of the preference parameters. We report results for values of risk aversion between one and five and for values of the elasticity of intertemporal substitution between 0.2 and 0.8, since this is the range

\(^{18}\)There are several reasons for eliminating these households. First, there are very few observations within each age group beyond age 75. Second, for most of these households the values of \( h_{it} \) are equal to zero. Third, this is consistent with the estimation procedure used for the labor income process.

\(^{19}\)Net income increases during the first years of retirement because the housing expenditures (\( h_t \)) are still positive and decreasing towards zero.
that we will consider in the remaining part of the paper, and it is consistent with existing empirical evidence (see discussion in IV.C.1). The top panel considers the first adult years $20 \leq t \leq 35$ during which wealth accumulation is mostly driven by the precautionary savings motive. As a result, the optimal consumption to wealth ratio is significantly more affected by prudence than by the elasticity of intertemporal substitution (EIS). Since the more risk averse investors are also the more prudent ones, the consumption to wealth ratio is a decreasing function of risk aversion. For very low values of risk aversion (close to 1) $C/X$ converges to the 100 percent limit imposed by the borrowing constraint.

The second panel of Table II summarizes the remaining pre-retirement period $(36 \leq t \leq 65)$, during which savings are now determined by the preference for low-frequency consumption smoothing, while the bottom panel reports the results for the retirement period $(66 \leq t \leq 100)$. The results are qualitatively identical in both cases. The optimal consumption to wealth ratio is driven by the trade-off between the (endogenous) expected return on invested wealth and the discount rate, combined with the household’s sensitivity to these incentives (the EIS). The less risk averse households invest a larger fraction of their portfolio in stocks, and therefore the expected return on their invested wealth is higher. Thus, since for a given $\psi$ less than one the income effect dominates the substitution effect, they will have higher consumption to wealth ratios. The same intuition explains why, for a given $\rho$, $C/X$ is a decreasing function of the EIS. However, for both the highest and lowest values of $\rho$ that we consider, this pattern becomes weaker. In the first case the expected return on invested wealth is very close to the discount rate and the consumption to wealth ratio is almost independent of $\psi$. In the second case $C/X$ is close to the 100 percent limit given by the borrowing constraint.

From the results in tables II we can conclude that, for the range of values that we consider, the consumption to wealth ratio is a decreasing function of both $\rho$ and $\psi$, at every stage of the life cycle.

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20 The results are qualitatively similar to the ones obtained by Campbell and Viceira (1999) in the context of an infinite-horizon portfolio choice model without labor income and liquidity constraints.

21 As risk aversion increases further (not reported), the return on the portfolio falls below the discount rate and the consumption to wealth ratio becomes an increasing function of the EIS.
B Asset Allocation

Figure 2.2 graphs the unconditional mean asset allocation in equities ($\pi_t$) for the same preference parameters as in figure 2.1 ($\rho$ equal to five, $\psi$ equal to 0.2, and $b = 2.5$). Even though earnings risk is uninsurable, cash is a closer substitute for future labor income than stocks (see Heaton and Lucas (1997)). Young households are “overinvested in their human capital” and view this non-tradeable asset as an implicit riskless asset in their portfolio. Given that the holdings of this relatively riskless asset are larger in the early part of the life cycle, all young households participate in the stock market and they allocate most of their financial wealth to stocks.\(^{22}\) As retirement approaches, and financial wealth increases relative to the present value of future labor income, agents start investing in cash. When retirement savings is at its peak, more than 50 percent of total wealth is now being invested in the riskless asset.

During retirement both future labor income (the present value of the pension transfers) and financial wealth are falling, so that the optimal asset allocation is determined by the relative speed at which these two decrease. Naturally this depends both on the discount rate (adjusted for the survival probabilities) and the strength of the bequest motive. Given our parameter values, during most of the retirement period, future labor income and wealth decay at similar rates, and as a result the share of wealth invested in stocks remains approximately constant.\(^{23}\)

IV Results with the Fixed Participation Cost

A Baseline case

We start by reporting the results for the baseline preference parameters ($\rho$ equal to five, $\psi$ equal to 0.2, and $b = 2.5$). In the next sections we consider different values.

\(^{22}\)During the very first years of adult life households hold a small fraction of their wealth in cash since the present value of future labor income is actually still increasing.

\(^{23}\)Except during the last years, when most households have very little financial wealth left.
A.1 Participation decision and asset allocation

The participation decision is determined by four factors. First, it is an increasing function of wealth accumulation. Intuitively, households that accumulate more wealth over the life cycle have a stronger incentive to enter the stock market. Second, for the same level of wealth accumulation, participation is a positive function of the optimal share of wealth invested in stocks. Third, since $F$ is an one-time cost, participation is also a positive function of the investment horizon. Fourth, since the cost must be paid at the time of entry, the likelihood of participating in the stock market is a negative function of current marginal utility.

Figure 3.1 shows the stock market participation rate for the baseline preference parameters. Since young households are liquidity constrained, their marginal utility is extremely high and as a result they do not participate in the stock market until sufficient wealth has been accumulated. As we can see from Figure 3.1, given these preference parameters, this happens very fast and by age 25 the participation rate is almost 100 percent. As a result, the average life cycle profiles of wealth accumulation, consumption and equity shares are almost identical to the ones obtained without the fixed cost (reported in figures 2.3 and 2.4), and are therefore omitted here.

A.2 Wealth distributions

Figure 3.2 plots the evolution of the distributions of cash on hand for the two types of agents: stock market participants and non-participants at age 30, still with $\rho$ equal to five and $\psi$ equal to 0.2. There is a pronounced spike around the normalized cash on hand level of 0.75; beyond that level, stock market participation becomes optimal and the two distributions overlap for a small interval, mostly representing the incurrence of the fixed entry cost. Figure 3.3 plots the distributions of cash on hand for ages 50 for both types of agents. Conditional on age, the distribution of cash on hand for stock holders has a much higher variance than the wealth distribution for the households that have not participated in the stock market.
A.3 Sensitivity analysis

Bequest motive

Next we perform some sensitivity analysis with respect to the importance of the bequest motive. Figure 3.4 plots wealth accumulation for different values of the parameter $b$, while figure 3.5 plots the corresponding conditional asset allocations. A stronger bequest motive increases wealth accumulation at every stage of the life cycle, and the effect is strongest during retirement. The increase in wealth accumulation leads to a modest reduction in the share of equity investment during working life. Since both of these effects have a fairly modest impact until retirement, the implied participation rates are not significantly affected and therefore we do not report them. During retirement, an increase in the bequest motive decreases the speed at which wealth is being drawn down, and leads to a higher ratio of financial wealth to labor income. As a result, for a given age, a stronger (weaker) bequest motive decreases (increases) the optimal equity share.

Correlation between stock returns and labor income shocks

As mentioned before, the empirical evidence on the magnitude of the correlation coefficients between stock returns and the different labor income shocks (transitory and permanent) is mixed. In our baseline calibration we have assumed $\phi_N$ equal to 0.15 and $\phi_U$ equal to 0.0, following the estimation of Campbell et al. (2001). In figure 3.6 we now check if the results are sensitive to these values. We only report the asset allocation decisions, since the participation decision is almost identical in all cases.

Campbell et al. (2001) do not actually estimate $\phi_U$ equal to 0.0, they just assume it. Therefore, in our first experiment we allow for a positive correlation between stock returns and the transitory labor income shocks, in particular we consider $\phi_N$ equal to 0.15 and $\phi_U$ equal to 0.1. The results are very similar to the ones obtained for the baseline case. In the second experiment we now set $\phi_N = 0.0$ and assume that the correlation is instead driven exclusively by the transitory shocks, thus setting $\phi_U$ equal to 0.15. We again obtain results that are extremely close to our baseline case. The share invested in equities is higher but only marginally so. Finally, we consider the case in which there is no correlation between stock returns and labor income shocks. Only in this case do we find a visible difference.
relative to the benchmark calibration, as investors now allocate a higher fraction of their wealth to equities.

**Initial wealth distribution**

So far we have assumed that all households start at age 20 with zero initial wealth. Table III reports summary statistics for the wealth distribution for households of age 20 (or lower) from the *Survey of Consumer Finances* (details given in Appendix C). When we use this distribution as the initial condition in our model, with the exception of the first few years, the wealth profiles are virtually indistinguishable as therefore we do not report them.\(^{24}\) This result occurs because young households are liquidity constrained and they therefore prefer to consume all of this additional wealth rather than save it.

**Housing expenditures**

In our baseline calibration we assume that housing expenditures constitute a fixed proportion of labor income. We now allow for a stochastic component in this ratio. More precisely, disposable income is now given by \((1 - \tilde{h}_t)Y_{i,t+1}\) where

\[
\tilde{h}_t = h_t \exp(\varepsilon^h_t) \tag{17}
\]

and \(\varepsilon^h_t\) follows a normal distribution with zero mean and variance \(\sigma^2_{\varepsilon^h}\).

In this experiment we set \(\sigma^2_{\varepsilon^h}\) equal to 0.25. This effectively corresponds to an increase in the level of background risk, and it is equivalent to an increase in the variance of the transitory labor income shocks. We find that the results are quite similar to the baseline case and therefore we do not report them. The increase in background risk reduces the willingness to invest in stocks, but since these are transitory shocks the effect is not very large. For the same reason, the wealth accumulation and the participation rate are almost unaffected.

**B Changing risk aversion and the impact of background risk**

The stock market participation rate implied by the baseline parameters is counterfactually high. In this section we explore the model’s ability to produce more realistic results by

\(^{24}\)In the implementation we truncate the distribution from the *SCF* at its 90th percentile.
considering different preference parameter values. As mentioned before, the participation decision is an increasing function of both wealth \((X)\) and the optimal share of wealth invested in risky assets \((\alpha)\). Decreasing risk aversion increases the optimal share invested in stocks, but as shown in section III.A, it also decreases wealth accumulation at every stage of the life cycle. Therefore, the impact on the participation decision resulting from changes in risk aversion, will depend on which of these two effects dominates.

### B.1 Wealth accumulation

We start by decreasing \(\rho\) from five to two, while maintaining the power utility assumption, thus increasing the elasticity of intertemporal substitution \((\psi)\) to 0.5. In figure 4.1 we plot the wealth accumulation for this case and for the baseline parameter values \((\rho\) equal to five, \(\psi\) equal to 0.2 and \(b = 2.5\)). As expected, wealth accumulation is significantly reduced at every stage of the life cycle. As previously shown in section III.A (see Table II), the average consumption to wealth ratio is now 86 percent for the age group 20 – 35, and 35 percent for the the age group 36 – 65, as opposed to 66 percent and 19 percent respectively. However, from the results in table II, we know that if we depart from power utility and decrease both risk aversion \((\rho)\) and the EIS \((\psi)\) simultaneously, this will significantly reduce wealth accumulation. Consider then decreasing \(\rho\) to two, but now keeping \(\psi\) at 0.2. The consumption wealth ratio for the first age group is not significantly affected (90 percent instead of 86 percent) since, at this stage of the life cycle, savings are essentially driven by prudence (which remains constant). However, for the second age group, wealth accumulation is determined mostly by the elasticity of intertemporal substitution. As a result, the average consumption to wealth ratio is now almost doubled, increasing from 35 percent to 67 percent. As shown in figure 4.1, this leads to a very significant reduction in life cycle wealth accumulation.

### B.2 Stock Market Participation Rates and Asset Allocation

Figure 4.2 plots the participation rates for different values of risk aversion, with the elasticity of intertemporal substitution equal to 0.2. Given the large differences in wealth accumulation, it is not surprising that the wealth effect dominates with respect to the participation decision.
The less prudent households save less and as a result their participation rate is smaller. While almost all high prudence households have already paid the fixed cost by age 25, only 75 percent of the households with $\rho$ equal to two have done so, although by age 35 even all of these investors have already paid the fixed cost as well. However, from the less risk averse households (i.e. $\rho$ equal to 1.2), only a very small fraction (less than 20 percent) will ever invest in stocks. On the other hand, as shown in figure 4.3, the reduction in risk aversion generates counterfactually high equity holdings for those investors that have paid the fixed cost.\footnote{For $\rho = 1.2$ the conditional equity share is always 100 percent, and therefore it is not included in the figure.}

### B.3 The impact of background risk

The previous results illustrate one important trade-off generated by the level of background risk. When faced with more background risk (for example, due to more labor income risk, consumption risk, or housing/mortgage risk) agents will invest a smaller fraction of their financial wealth in risky assets. However, they will also accumulate a larger buffer stock of wealth, thus having a stronger incentive to enter the stock market. We have considered three different experiments in which we have increased the investor’s background risk. In the first two we have assumed a higher variance of respectively transitory and permanent labor income shocks, and in the third we have included a positive probability of a disastrous labor income shock. Figure 4.4 shows the results for the case of the first experiment, where the variance of the transitory labor income shocks was increased by a factor of three.\footnote{The results for the other two cases are qualitatively identical, and they are available upon request.} As expected, background risk crowds-out stock holdings and households invest a smaller fraction of their portfolio in equities. However, they also increase their buffer stock of wealth, and as a result the stock market participation rate is higher than before.
C Asset allocation and participation rates with preference heterogeneity

C.1 Matching participation rates and conditional asset allocations

Given our previous results, we can simultaneously match stock market participation rates and asset allocation conditional on participation, with moderate degrees of risk aversion, if we allow for preference heterogeneity. Households with very low risk aversion and low $EIS$ smooth idiosyncratic earnings shocks with a small buffer stock of assets, and most of them never invest in equities (thus behaving as in the Deaton (1991) infinite horizon model). This seems to describe adequately the behavior of a large fraction of the U.S. population that retires without significant financial assets (and does not participate in the stock market). Within the low $EIS$ and low risk aversion group, only a small fraction owns stocks, and they do so only as they get close to retirement. On the other hand, investors with high prudence and high $EIS$ are the ones that participate in the stock market from early on, since they accumulate more wealth and therefore have a stronger incentive to pay the fixed cost. Therefore, the marginal stockholders are (endogenously) more risk averse and as a result they do not invest their portfolios fully in stocks.

In this final section we try to evaluate how much heterogeneity we need to match the data. In other words, can the model consistently explain the two facts for a plausible distribution of preference parameters across the population? Table $IV$ reports participation rates and average equity shares for stock market participants, for different distributions, and compares them with the empirical evidence from the SCF. We first consider a 50 percent split between investors with both low risk aversion and low $EIS$ ($\rho$ equal to 1.2 and $\psi$ equal to 0.2), and investors with moderate risk aversion and moderate $EIS$ ($\rho$ equal to five and $\psi$ equal to 0.5). The model delivers a participation rate of 52.1 percent and an equity share of 54.5 percent for stock market participants (Case one in table $IV$), which matches fairly well with the empirical evidence reported in section I (and summarized in the first row of table $IV$).

It is important to mention that this form of heterogeneity is consistent with the existing empirical evidence. Attanasio, Banks and Tanner (2002) show that the CRRA coefficient is much higher (thus much lower $EIS$) for non-stockholders than for stockholders. Vissing-
Jørgensen (2002a) focusses on this distinction and argues that “accounting for limited asset market participation is crucial for obtaining consistent estimates of the EIS” (p. 827). Vissing-Jørgensen then obtains estimates of the EIS greater than 0.3 for risky asset holders, while for the remaining households the EIS estimates are small and insignificantly different from zero. Vissing-Jørgensen and Attanasio (2003) further stress that loosening the link between risk aversion and intertemporal substitution can generate implications about the covariance of stock returns and individual consumption growth for stockholders that are not rejected in the data. They offer risk aversion estimates for stockholders at around five-ten and EIS estimates around one. Overall, the existing estimates of EIS and risk aversion are consistent with the values that we use in this paper.

C.2 Life cycle profiles

We now report the life cycle profiles of stock market participation and asset allocation implied by the model. As argued in section I, in the data these profiles are not very robust to specific assumptions about cohort or time effects (see figures 1.1 to 1.4). As a result, in this paper we have mostly focused on life cycle averages.

Figure 5.1 plots the stock market participation rate implied by the model for different age groups, together with the corresponding numbers from the SCF (see appendix C for details), while figure 5.2 does the same but now for the average asset allocation of stock market participants. The participation rates are extremely similar, with the largest difference occurring at retirement when the participation rate in the data declines while it remains constant in the model. However, as shown in figure 1.2, this is exactly one of the results that is not robust to the assumption of cohort dummies versus time dummies. With respect to the asset allocation decisions, we do observe a more significant difference, in this case for young households. In the model these agents invest a significant fraction of their portfolio in equities while in the data, regardless of the controls, this does not happen.
C.3 Sensitivity analysis and robustness

So far we have assumed that households start at age 20 with zero initial wealth since we have seen in section IV.A.3 that, for $\rho$ equal to five, using the initial wealth distribution calibrated from the SCF does not affect the results. However, this might not be the case for investors with low risk aversion and low $EIS$ since they save very little. By giving these households some positive initial wealth, we are likely to see an increase in stock market participation rates. In the third row of table IV (Case two) we show that this effect is not too large, and we can replicate the previous results by considering a slightly lower value of risk aversion ($1.1$).\(^{27}\)

It is important to point out that we do not need to assume a very low value of the $EIS$ to generate large non-participation, since we can compensate for a higher $\psi$ by decreasing risk aversion even further. This is shown in the fourth row of table IV (Case three), where we fix the $EIS$ coefficient equal to 0.5 for both types of investors. To reproduce the results in the first panel, we find that we need to decrease $\rho$ to 1.07 for the less risk averse group.

Given our previous discussion, we know that households with risk aversion between 1.5 and four will tend to participate in the stock market from early on, and invest almost all of their wealth in stocks. Naturally, it is not reasonable to assume that the distribution of coefficients of risk aversion mysteriously collapses to the two extremes that we have previously considered (1.2 and five). In the fifth row of table IV (we now consider a smoother distribution, with $\rho$ ranging from one to five (Case four). It is important to point out that this is not a uniform distribution, as there is a slightly higher fraction of less risk averse households. If we want to match both facts simultaneously, with a (relatively) smooth distribution, we need it to exhibit some negative skewness. As predicted, both the participation rate and the equity share are now higher than before but not significantly so. The equity share is now 57 percent, while the participation rate is also equal to 57 percent, numbers that are still extremely close to the empirical evidence (row 1).

\(^{27}\)Alternatively we could consider a lower value of the $EIS$. With $\psi = 0.1$ we would again obtain very similar results.
D Wealth Distribution

In this section we compare the wealth accumulation predicted by the model, with the empirical evidence in the SCF. Given the (exogenous) differences in the preference parameters and the (endogenous) differences in the participation decision, our model generates a large degree of heterogeneity in wealth accumulation. To illustrate, we compare both median wealth accumulation and the extremes of the distribution (10th and 90th percentiles) to see if the model generates the degree of heterogeneity observed in the SCF. We divide households in the three usual age groups: buffer stock savers (20 – 35), retirement savers (36 – 65) and retirees (over 66).

The results are shown in table V. The model can replicate the low wealth accumulation patterns of the poorer households in the data. Households with the lowest income realizations tend not to participate in the stock market and accumulate very little wealth over the life cycle. This is consistent with the results in Hubbard, Skinner and Zeldes (1995) who illustrate in a similar model how the presence of social insurance (pensions) can crowd out private saving over the life cycle for the poorest quintile of the wealth distribution. Nevertheless, in the SCF these households still accumulate some non-negligible wealth during retirement, something that does not happen in the model. For the median household, the model does quite well early in life, it overshoots for the second age group, and undershoots at retirement. Finally, at the high-end of the distribution we can generate extremely large wealth accumulation, although not quite as high as in the data. This difference is most significant during the retirement period and early in life. Overall the degree of heterogeneity in the wealth distribution is comparable to the one observed in the data. The model consistently generates low wealth accumulation at retirement, which would suggest the presence of a stronger bequest motive but, as shown in section IV.A.3, a stronger bequest motive also increases wealth accumulation at mid-life (figure 3.4).

In the final panel of table V we simulate the model with the initial distribution of wealth calibrated from the SCF. The results are almost identical to the previous ones: we only observe a minor increase in wealth accumulation at the 90th percentile.
V Conclusion

In this paper we present a life cycle asset allocation model with realistically calibrated uninsurable labor income risk, that provides an explanation for two very important empirical observations: low stock market participation rates in the population as a whole, and moderate equity holdings for stock market participants. We do not rely on high values of risk aversion, or on extreme assumptions about background risk.

In our model households with very low risk aversion and low $EIS$, accumulate very little wealth and as a result (most of them) never invest in equities. On the other hand, the more prudent investors are the ones that participate in the stock market from early on, as they accumulate more wealth and therefore have a stronger incentive to pay the fixed entry cost. Therefore, the marginal stockholders are (endogenously) more risk averse and as a result they do not invest their portfolios fully in stocks. On the negative side the model still counterfactually predicts that, young households that have already paid the participation cost, will invest most of their portfolio in equities.
Appendix A: Numerical Solution Method

We first simplify the solution by exploiting the scale-independence of the maximization problem and rewriting all variables as ratios to the permanent component of labor income ($P_{it}$). The laws of motion and the value function can then be rewritten in terms of these normalized variables, and we use lower case letters to denote them (for instance, $x_{it} \equiv \frac{x_{it}}{P_{it}}$). This allows us to reduce the number of state variables to three; one continuous state variable (cash on hand, $x_{it}$) and two discrete state variables (age, $t$, and participation status, whether the fixed cost has been paid or not). We discretize the state-space along the cash on hand dimension (the only continuous state variable), so that the relevant policy functions can now be represented on a numerical grid.

We solve the model using backward induction. For every age $t$ prior to $T$, and for each point in the state space, we optimize using grid search. So we need to compute the value associated with each level of consumption, the decision to pay the fixed cost, and the share of liquid wealth invested in stocks. From the Bellman equation, these values are given as current utility plus the discounted expected continuation value ($E_t V_{t+1}(..)$), which we can compute once we have obtained $V_{t+1}$. In the last period the policy functions are determined by the bequest motive and the value function corresponds to the bequest function, regardless of whether the fixed cost has been paid before or not. This gives us the terminal condition for our backward induction procedure. We perform all numerical integrations using Gaussian quadrature to approximate the distributions of the innovations to the labor income process and the risky asset returns. We evaluate the value function, for points which do not lie on state space grid, using a cubic spline interpolation.

Once we have computed the value of all the alternatives we just pick the maximum, thus obtaining the policy rules for the current period ($S_t$ and $B_t$). At each point of the state space, the participation decision is computed by comparing the value function conditional on having paid the fixed cost (adjusting for the payment of the cost itself) with the value function conditional on non-payment. Substituting these decision rules in the Bellman equation we obtain this period’s value function ($V_t(..)$), which is then used to solve the previous period’s maximization problem. This process is iterated until $t = 1$. 

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Appendix B: Computing the Transition Distributions

To find the distribution of cash on hand, we first compute the relevant optimal policy rules; bond and stock policy functions for stock market participants and non-participants and the \( \{0,1\} \) participation rule as a function of cash on hand. Let \( b^I(x) \) and \( s^I(x) \) denote respectively the bonds and stock policy rules for individuals participating in the stock market, and let \( b^O(x) \) be the savings decision for the individual out of the stock market. We assume that households start their working life with zero liquid assets. During working life, for the households that have not paid the fixed cost, the evolution of normalized cash on hand is given by\(^{28}\)

\[
x_{t+1} = \left[ b^O(x_t) R_f \right] \left\{ \frac{P_t}{P_{t+1}} \exp(f(t, Z_t)) \right\} + (1 - h_{t+1}) U_{t+1}
= w \left( \frac{x_t}{P_t}, \frac{P_t}{P_{t+1}} \right) \frac{\exp(f(t, Z_t))}{\exp(f(t+1, Z_{t+1}))} + (1 - h_{t+1}) U_{t+1}
\tag{A1}
\]

where \( w(x) \) is defined by the last equality and is conditional on \( \left\{ \frac{P_t}{P_{t+1}} \right\} \) and the deterministically evolving \( \frac{\exp(f(t, Z_t))}{\exp(f(t+1, Z_{t+1}))} \). Denote the transition matrix of moving from \( x_j \) to \( x_k \),\(^{29}\) conditional on not having paid the fixed cost as \( T^O_{kj} \). Let \( \Delta \) denote the distance between the equally spaced discrete points of cash on hand. The random permanent shock \( P_t \) is discretized using Gaussian quadrature with \( H \) points:

\[
\frac{P_t}{P_{t+1}} = \{N_m\}_{m=1}^{m=H} \cdot T^O_{kj} = \Pr(x_{t+1}=k|x_t=j)
\]

is found using\(^{30}\)

\[
\sum_{m=1}^{m=H} \Pr \left( x_{t+1}|x_t, \frac{P_t}{P_{t+1}} = N_m \right) \ast \Pr \left( \frac{P_t}{P_{t+1}} = N_m \right)
\tag{A2}
\]

Numerically, this probability is calculated using

\[
T^O_{kjm} = \Pr \left( x_k + \frac{\Delta}{2} \geq x_{t+1} \geq x_k - \frac{\Delta}{2} | x_t = x_j, \frac{P_t}{P_{t+1}} = N_m \right)
\tag{A3}
\]

Making use the approximation that for small values of \( \sigma_u^2 \), \( U \sim N(\exp(\mu_u + .5 \cdot \sigma_u^2), (\exp(2 \cdot \mu_u + (\sigma_u^2)) \ast (\exp(\sigma_u^2) - 1))) \), and denoting the mean of \( (1 - h_t)U \) by \( \overline{U} \) and its standard

\(^{28}\)To avoid cumbersome notation, the subscript \( i \) that denotes a particular individual is omitted in what follows.

\(^{29}\)The normalized grid is discretized between \( (x_{\min}, x_{\max}) \) where \( x_{\min} \) denotes the minimum point on the equally spaced grid and \( x_{\max} \) the maximum point.

\(^{30}\)The dependence on the deterministically evolving \( \frac{\exp(f(t, Z_t))}{\exp(f(t+1, Z_{t+1}))} \) is implied and is omitted from what follows for expositional clarity.
deviation by \( \sigma \), the transition probability conditional on \( N_m \) equals
\[
T_{kjm}^O = \Phi \left( \frac{x_k + \frac{\Delta}{2} - w(x_t|N_m) - \mathcal{U}}{\sigma} \right) - \Phi \left( \frac{x_k - \frac{\Delta}{2} - w(x_t|N_m) - \mathcal{U}}{\sigma} \right)
\] (A4)
where \( \Phi \) is the cumulative distribution function for the standard normal. The unconditional probability from \( x_j \) to \( x_k \) is then given by
\[
T_{kj}^O = \sum_{m=1}^{m=H} T_{kjm}^O \Pr(N_m)
\] (A5)
Given the transition matrix \( T^O \) (letting the number of cash on hand grid points equal to \( J \), this is a \( J \) by \( J \) matrix; \( T_{kj}^O \) represents the \( \{kth,jth\} \) element), the next period probabilities of each of the cash on hand states can be found using
\[
\pi^O_{kt} = \sum_j T_{kj}^O \pi^O_{jt-1}
\] (A6)
We next use the vector \( \Pi^O_t \) (this is a \( J \) by 1 vector representing the mass of the population out of the stock market at each grid point; \( \pi^O_{kt} \) represents the \( \{kth\} \) element at time \( t \)) and the participation policy rule to determine the percentage of households that optimally choose to incur the fixed cost and invest in risky assets. This is found by computing the sum of the probabilities in \( \Pi^O_t \) for which \( x > x^* \), \( x^* \) being the trigger point that causes participation (\( x^* \) is determined endogenously through the participation decision rule). These probabilities are then deleted from \( \Pi^O_t \) and are added to \( \Pi^I_t \), appropriately renormalizing both \( \{\Pi^O_t, \Pi^I_t\} \) to sum to one. The participation rate (\( \theta_t \)) can be computed at this stage as
\[
\theta_t = \theta_{t-1} + (1 - \theta_{t-1}) \sum_{x_j > x^*} \pi^O_{tj}
\] (A7)
The same methodology (but with more algebra and computations) can then be used to derive the transition distribution for cash on hand conditional on having paid the fixed cost, \( T^I_t \). The corresponding normalized cash on hand evolution equation is
\[
x_{t+1} = \left[ b(x_t)R^f + s(x_t)R^S_{t+1} \right] \left\{ \frac{P_t}{P_{t+1}} \frac{\exp(f(t, Z_t))}{\exp(f(t + 1, Z_{t+1}))} \right\} + (1 - h_{t+1})U_{t+1}
\]
\[
= w \left( x_t|R^S_{t+1} \frac{P_t}{P_{t+1}} \right) + (1 - h_{t+1})U_{t+1}
\] (A8)
where \( w(x) \) is now conditional on \( \{R_{t+1}^S, \frac{P_t}{P_{t+1}}\} \). The random processes \( R_{t+1}^S \) and \( \frac{P_t}{P_{t+1}} \) are discretized using Gaussian quadrature with \( H \) points: \( R_{t+1}^S = \{R_i^S\}_{i=1}^H \) and \( \frac{P_t}{P_{t+1}} = \{N_n\}_{n=1}^H \).

\[
T_{kj}^l = \Pr(x_{t+1}=k|x_t=j) \text{ is obtained from}
\]

\[
\sum_{l=1}^H \sum_{n=1}^H \Pr \left( x_{t+1} = x_t, R_{t+1}^S = R_i^S, \frac{P_t}{P_{t+1}} = N_n \right) \cdot \Pr(R_i^S) \cdot \Pr(N_n)
\]

(A9)

where \( \Pr(R_i^S) \) and \( \Pr(N_n) \) stand respectively for \( \Pr(R_{t+1}^S = R_i^S) \) and \( \Pr\left( \frac{P_t}{P_{t+1}} = N_n \right) \), and where the independence between \( \frac{P_t}{P_{t+1}} \) and \( R_{t+1}^S \) was used. Numerically, this probability is calculated using

\[
T_{kj}^l = \Pr \left( x_k + \frac{\Delta^2}{2} \geq x_{t+1} \geq x_k - \frac{\Delta^2}{2}, x_t = x_j, \frac{P_t}{P_{t+1}} = N_n, R_{t+1}^S = R_i^S \right)
\]

(A10)

The transition probability conditional on \( N_n, R_i^S \) and \( R_m^B \) equals

\[
T_{kj}^l = \Phi \left( \frac{x_k + \frac{\Delta^2}{2} - w(x_t|N_n, R_i^S) - \bar{U}}{\sigma} \right) - \Phi \left( \frac{x_k - \frac{\Delta^2}{2} - w(x_t|N_n, R_i^S) - \bar{U}}{\sigma} \right)
\]

(A11)

The unconditional probability from \( x_j \) to \( x_k \) is then given by

\[
T_{kj}^l = \sum_{l=1}^H \sum_{n=1}^H T_{kjln}^l \cdot \Pr(R_i^S) \cdot \Pr(N_n)
\]

(A12)

Given the matrix \( T^l \), the probabilities of each of the states are updated by

\[
\pi_{kt+1}^I = \sum_j T_{kj}^l \cdot \pi_{jt}^I
\]

(A13)

31 The dependence on the non-random earnings component is omitted to simplify notation.

32 The methodology can be applied for an arbitrary correlation structure using the Choleski decomposition of the variance-covariance matrix of the innovations.
Appendix C: Survey of Consumer Finances Data

The *Survey of Consumer Finances (SCF)* is probably the most comprehensive source of data on U.S. household assets. The SCF uses a two-part sampling strategy to obtain a sufficiently large and unbiased sample of wealthier households (the rich sample is chosen randomly using tax reports). To enhance the reliability of the data, the SCF makes weighting adjustments for survey non-respondents; these weights were used in computing the values reported in the tables. The specific names in the codebook for the variables used are given below.

We construct a measure of labor income that matches as closely as possible the process for $Y_{it}$ (earnings) in the text. We therefore define labor income as the sum of wages and salaries (X5702), unemployment or worker’s compensation (X5716) and Social Security or other pensions, annuities, or other disability or retirement programs (X5722). Liquid wealth is variable FIN in the publicly available SCF data set, to which home equity was added. Variable FIN is made up of LIQ (all types of transaction accounts (checking, saving, money market and call accounts)), CDS (certificates of deposit), total directly-held mutual funds, stocks, bonds, total quasi-liquid financial assets (the sum of IRAs, thrift accounts, and future pensions), savings bonds, the cash value of whole life insurance, other managed assets (trusts, annuities and managed investment accounts in which the household has equity interest) and other financial assets: includes loans from the household to someone else, future proceeds, royalties, futures, non-public stock, and deferred compensation. We define home equity as the value of the home less the amount still owed on the first and 2nd/3rd mortgages and the amount owed on home equity lines of credit. This definition of wealth is consistent with both the definition in Hubbard, Skinner and Zeldes (1995) and Heaton and Lucas (2000).

Financial assets invested in the risky asset can either be directly-held stock or stock mutual funds or amounts of stock in retirement accounts. We follow the procedures the SCF uses to construct this number for each household (variable EQUITY). Specifically, this is done by computing the full value of stocks, adding the full value if an asset is described as a stock mutual fund, and half the value if the asset refers to a combination of mutual funds. To this, IRAs/Keoghs invested in stock are computed by adding the full value if mostly invested
in stock, half the value if split between stocks/bonds or stocks/money market, and one third of the value if split between stocks/bonds/money market. We also add other managed assets with equity interest (annuities, trusts, MIAs) by adding the full value if mostly invested in stock, half the value if split between stocks/MFs & bonds/CDs, or “mixed/diversified” and one third of the value if “other”. We also add thrift-type retirement accounts invested in stock: the full value if mostly invested in stock and half the value if split between stocks and interest earning assets. Stock market participation is then determined by checking whether the full value of stocks (EQUITY) is greater than zero (variable HEQUITY).

We construct the share of wealth in stocks conditional on HEQUITY being positive as (EQUITY)/(FIN) where all the variables have been defined above.
References


Constantinides, George, 1986, Capital Market Equilibrium with Transaction Costs, *Jour-


Table I: Regression of the ratio of housing expenditures to labor income \((he_{it})\), on age polynomials and time dummies. The data is taken from the Panel Study of Income Dynamics from 1976 until 1993. For each household, in each year, we compute the ratio of annual mortgage payments plus rent payments relative to annual labor income, and regress this ratio against a constant a cubic polynomial of age (where age is defined as the age of the head of the household) and time dummies. We eliminate all observations with age greater than 75.

<table>
<thead>
<tr>
<th></th>
<th>coefficient</th>
<th>t-stat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>0.703998</td>
<td>5.47</td>
</tr>
<tr>
<td>Age</td>
<td>−0.0352276</td>
<td>−3.70</td>
</tr>
<tr>
<td>Age(^2)</td>
<td>0.0007205</td>
<td>3.17</td>
</tr>
<tr>
<td>Age(^3)</td>
<td>−0.0000049</td>
<td>−2.84</td>
</tr>
<tr>
<td>adj. R(^2)</td>
<td>0.025</td>
<td></td>
</tr>
</tbody>
</table>
Table II: Average Consumption-Wealth Ratio ($C/X$) implied by the no-fixed cost model, for different values of both the coefficient of risk aversion ($\rho$) and the elasticity of intertemporal substitution ($\psi$), and for different age groups.

<table>
<thead>
<tr>
<th>$\psi$</th>
<th>$\rho$</th>
<th>age 20 until age 35</th>
<th>age 36 until age 65</th>
<th>age 66 until age 100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$= 0.8$</td>
<td>1</td>
<td>98% 99% 99%</td>
<td>98% 99% 99%</td>
<td>100% 100% 100%</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>87% 92% 93%</td>
<td>43% 88% 94%</td>
<td>88% 100% 100%</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>76% 86% 90%</td>
<td>18% 35% 67%</td>
<td>25% 71% 97%</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>61% 67% 75%</td>
<td>14% 18% 27%</td>
<td>23% 29% 59%</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>55% 60% 66%</td>
<td>13% 16% 19%</td>
<td>25% 26% 47%</td>
</tr>
<tr>
<td>$= 0.5$</td>
<td>1</td>
<td>98% 99% 99%</td>
<td>98% 99% 99%</td>
<td>100% 100% 100%</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>87% 92% 93%</td>
<td>43% 88% 94%</td>
<td>88% 100% 100%</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>76% 86% 90%</td>
<td>18% 35% 67%</td>
<td>25% 71% 97%</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>61% 67% 75%</td>
<td>14% 18% 27%</td>
<td>23% 29% 59%</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>55% 60% 66%</td>
<td>13% 16% 19%</td>
<td>25% 26% 47%</td>
</tr>
<tr>
<td>$= 0.2$</td>
<td>1</td>
<td>98% 99% 99%</td>
<td>98% 99% 99%</td>
<td>100% 100% 100%</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>87% 92% 93%</td>
<td>43% 88% 94%</td>
<td>88% 100% 100%</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>76% 86% 90%</td>
<td>18% 35% 67%</td>
<td>25% 71% 97%</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>61% 67% 75%</td>
<td>14% 18% 27%</td>
<td>23% 29% 59%</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>55% 60% 66%</td>
<td>13% 16% 19%</td>
<td>25% 26% 47%</td>
</tr>
</tbody>
</table>
Table III: Wealth distribution (wealth to income ratios) for households with head aged 20 or less. The data are taken from the 2001 Survey of Consumer Finances (details in Appendix C). $X$ defines liquid wealth and $Y$ denotes labor or pension income.

<table>
<thead>
<tr>
<th>decile</th>
<th>10%</th>
<th>20%</th>
<th>30%</th>
<th>40%</th>
<th>50%</th>
<th>60%</th>
<th>70%</th>
<th>80%</th>
<th>90%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X/Y$</td>
<td>0.000</td>
<td>0.015</td>
<td>0.043</td>
<td>0.113</td>
<td>0.167</td>
<td>0.236</td>
<td>0.267</td>
<td>0.406</td>
<td>0.863</td>
</tr>
</tbody>
</table>

Table IV: Average stock market participation rate ($\overline{P}$) and average stock holdings for stock market participants ($\overline{\pi}_P$). The first row reports Data from the 2001 Survey of Consumer Finances (details in Appendix C), while the other four panels report the results from model for different distributions of investors. Case 1 assumes 2 groups of agents, ($\rho = 1.2$ and $\psi = 0.2$) and ($\rho = 5$ and $\psi = 0.5$), with 50% weight each. Case 2 also assumes 2 groups of agents, but now ($\rho = 1.1$ and $\psi = 0.2$) and ($\rho = 5$ and $\psi = 0.5$), with 50% weight each, and with the initial wealth distribution calibrated from the SCF. In Case 3 we have again two groups, ($\rho = 1.07$ and $\psi = 0.5$) and ($\rho = 5$ and $\psi = 0.5$), with 50% weight each. Finally, Case 4 considers 3 groups of agents, ($\rho = 1$ and $\psi = 0.2$) and ($\rho = 3$ and $\psi = 0.5$) and ($\rho = 5$ and $\psi = 0.5$), with weights 40%, 30% and 30% respectively.

<table>
<thead>
<tr>
<th></th>
<th>$\overline{P}$</th>
<th>$\overline{\pi}_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>51.94%</td>
<td>54.76%</td>
</tr>
<tr>
<td>Model (Case 1)</td>
<td>52.14%</td>
<td>54.48%</td>
</tr>
<tr>
<td>Model (Case 2)</td>
<td>50.36%</td>
<td>53.32%</td>
</tr>
<tr>
<td>Model (Case 3)</td>
<td>54.42%</td>
<td>56.24%</td>
</tr>
<tr>
<td>Model (Case 4)</td>
<td>56.98%</td>
<td>56.56%</td>
</tr>
</tbody>
</table>
Table V: Distribution of wealth to labor income ratios from the 2001 *Survey of Consumer Finances*, for different age groups (appendix C provides more details), and for two different versions of the model: with zero initial wealth and with the initial wealth distribution calibrated from the *Survey of Consumer Finances* (SCF). Results are shown for different age groups.

<table>
<thead>
<tr>
<th>Age Groups</th>
<th>20–35</th>
<th>36–65</th>
<th>≥65</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Data</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10&lt;sup&gt;th&lt;/sup&gt; percentile</td>
<td>0.002</td>
<td>0.071</td>
<td>0.371</td>
</tr>
<tr>
<td>median</td>
<td>0.287</td>
<td>2.170</td>
<td>7.931</td>
</tr>
<tr>
<td>90&lt;sup&gt;th&lt;/sup&gt; percentile</td>
<td>2.702</td>
<td>10.648</td>
<td>33.363</td>
</tr>
<tr>
<td><strong>Model (zero initial wealth)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10&lt;sup&gt;th&lt;/sup&gt; percentile</td>
<td>0.006</td>
<td>0.005</td>
<td>0.006</td>
</tr>
<tr>
<td>median</td>
<td>0.261</td>
<td>3.115</td>
<td>4.838</td>
</tr>
<tr>
<td>90&lt;sup&gt;th&lt;/sup&gt; percentile</td>
<td>0.748</td>
<td>8.184</td>
<td>17.539</td>
</tr>
<tr>
<td><strong>Model (Initial wealth from the SCF)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10&lt;sup&gt;th&lt;/sup&gt; percentile</td>
<td>0.006</td>
<td>0.005</td>
<td>0.006</td>
</tr>
<tr>
<td>median</td>
<td>0.263</td>
<td>3.116</td>
<td>4.839</td>
</tr>
<tr>
<td>90&lt;sup&gt;th&lt;/sup&gt; percentile</td>
<td>0.886</td>
<td>8.371</td>
<td>17.865</td>
</tr>
</tbody>
</table>
Figure 1.1 - Equity holdings as a fraction of total financial wealth, for stock market participants. The results are taken from Ameriks and Zeldes (2001), and they are obtained from OLS regressions with age dummies and, either time or cohort dummies. The data includes the 1989, 1992, 1995 and 1998 waves of the Survey of Consumer Finances.

Figure 1.2 - Stock market participation rate. The results are taken from Ameriks and Zeldes (2001), and they are obtained from Probit regressions with age dummies and, either time or cohort dummies. The data includes the 1989, 1992, 1995 and 1998 waves of the Survey of Consumer Finances.
Figure 2.1 – Life cycle profiles of consumption, income and wealth for the baseline preference parameters: coefficient of relative risk aversion equal to five, elasticity of intertemporal substitution equal to 0.2, and bequest motive equal to 2.5.

Figure 2.2 – Life cycle asset allocation for the baseline preference parameters: coefficient of relative risk aversion equal to five, elasticity of intertemporal substitution equal to 0.2, and bequest motive equal to 2.5.
Figure 3.1 - Stock market participation over the life cycle for the baseline preference parameters: coefficient of relative risk aversion equal to five, elasticity of intertemporal substitution equal to 0.2, and bequest motive equal to 2.5.

Figure 3.2 - Distributions for normalized cash on hand, at age 30, for the baseline preference parameters: coefficient of relative risk aversion equal to five, elasticity of intertemporal substitution equal to 0.2, and bequest motive equal to 2.5.
Figure 3.3 - Distributions for normalized cash on hand, at age 50, for the baseline preference parameters: coefficient of relative risk aversion equal to five, elasticity of intertemporal substitution equal to 0.2, and bequest motive equal to 2.5.

Figure 3.4 - Wealth accumulation for different values of the bequest parameter (b), and for the baseline preferences parameters: coefficient of relative risk aversion equal to five, elasticity of intertemporal substitution equal to 0.2, and bequest motive equal to 2.5.
Figure 3.5 - Asset allocation for stock market participants, for different values of the bequest parameter (b), and for the baseline preference parameters: coefficient of relative risk aversion equal to five, elasticity of intertemporal substitution equal to 0.2.

Figure 3.6 - Asset allocation for stock market participants, for different values of the correlation between stock returns and transitory (permanent) labor income shocks, denoted by corrt (corrp), with the baseline preferences parameters: coefficient of relative risk aversion equal to five, elasticity of intertemporal substitution equal to 0.2, and bequest motive equal to 2.5.
Figure 4.1 - Wealth accumulation over the life cycle, for different values of the preference parameters (coefficient of relative risk aversion and elasticity of intertemporal substitution) with bequest motive equal to 2.5.

Figure 4.2 - Stock market participation rate for different values the coefficient of relative risk aversion (RRA), with elasticity of intertemporal substitution equal to 0.2, and bequest motive equal to 2.5.
Figure 4.3 - Asset allocation for stock market participants, for different values of the coefficient of relative risk aversion (RRA), with elasticity of intertemporal substitution equal to 0.2, and bequest motive equal to 2.5.

Figure 4.4 - Stock market participation rate and share of wealth invested in stocks for different levels of background risk, with the baseline preferences: coefficient of relative risk aversion equal to five, elasticity of intertemporal substitution equal to 0.2, and bequest motive equal to 2.5.
Figure 5.1 - Stock market participation rate implied by the heterogeneous agent model and stock market participation rate from the 2001 wave of the *Survey of Consumer Finances*.

Figure 5.2 - Asset allocation for stock market participants rate implied by the heterogeneous agent model and asset allocation for stock market participants from the 2001 wave of the *Survey of Consumer Finances*.