Optimal Financial Integration and Security Design\textsuperscript{1}

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Optimal Financial Integration and Security Design

Abstract

We study two-period pure-exchange Capital Asset Pricing Model (CAPM) economies, for given degrees of incompleteness of financial markets and given degrees of restricted participation of agents in the markets. We characterize the optimal financial market structure of this economy, as well as efficient financial innovations consisting both of the introduction of new assets, as well as of the integration of segmented markets. We show that the optimal financial market structures maximize a simple criterion that captures the dispersion of the ‘betas’ of the participating agents. We find that the order in which innovations are introduced affects the optimality of the financial structure if the nature of restricted participation varies across the assets, suggesting that some coordination of the innovation process is likely to be desirable for integrating economies when their asset markets are segmented, as, for instance, is the case for European financial integration.
1 Introduction

The theoretical literature on financial innovation has successfully analyzed the effects of the introduction of new assets to span the uncertainty of agents’ endowments and production plans in economies with incomplete financial markets (see the surveys in Allen and Gale, 1994-a, and Duffie and Rahi, 1995). Such literature has however paid much less attention to restricted participation in financial markets and to the integration of segmented markets as a form of financial innovation.¹

Indeed, the empirical evidence for limited participation in financial markets is ample (see Blume and Friend, 1975, Blume and Zeldes, 1993, and Mankiw and Zeldes, 1991); and while financial innovations usually consist of the introduction of new financial instruments as in the case of derivative securities, many recent innovations, like the introduction of electronic trading, and of the practice of securitization, represent instances of integration of segmented markets.

Even theoretical studies of optimal financial innovations in international capital market have mostly dealt with the introduction of new assets (see for instance Athanasoulis and Shiller, 1999), rather than with the integration of segmented financial markets which is instead of great empirical relevance, as e.g., the European financial integration experience indicates.

In this paper we study financial innovations consisting both of (i) the introduction of new assets, as well as of (ii) the integration of segmented markets, and we study the interrelations between these two forms of innovation. We have two main objectives. We first want to characterize the optimal structure of financial assets and the optimal level of integration of markets in the presence of exogenous limits to the degree of financial market completeness and to the degree of market participation. (While exogenous in our set-up, such restrictions on the structure of financial markets are to be interpreted as due to transaction costs.) Furthermore, we aim at a characterization which can be easily taken to the data, to address simple but important questions about Which financial innovations generate the highest welfare gains? Which financial markets are the best to integrate? or The financial markets of which country are the best to integrate?

Our second objective consists in the analysis of the overall efficiency of financial innova-

¹Notable exceptions are Allen and Gale (1994-b), who study volatility of asset prices in limited participation regimes vis-a-vis full participation regimes; Diamond (1997), who proposes restricted participation as an explanation for the existence of financial intermediaries; Basak and Cuoco (1998), who develop a continuous-time pure-exchange economy with restricted participation in stock market; and Heaton and Lucas (1999), who show that an increase in stock market participation along with greater ability to diversify can at least partially explain observed price run-ups.
In particular, suppose that each innovation introduced in financial markets is optimal for the given existing financial market structure, and that financial innovations are introduced sequentially by different intermediaries. Does the financial market structure which results from such a sequence of financial innovations necessarily coincide with the optimal financial market structure? Is the order of financial innovation immaterial for welfare considerations in our class of economies?

If this is so, we say that the optimal financial asset structure is \textit{decentralizable}.\footnote{We do not model explicitly the process by which innovations enter the financial markets. For such kind of models, see Duffie and Jackson (1989) who take a volume-maximizing objective of intermediation with proportional transaction costs. Also, see Bisin (1998), Cuny (1993), Madan and Soubra (1991), Pesendorfer (1995), and Ross (1989), for economies with transaction costs; and Boot and Thakor (1993), Demarzo (1999), Duffie and Demarzo (1999), Glaeser and Kallal (1997), and Nachman and Noe (1994), for economies with asymmetric information. Further, the degree of market incompleteness as well as the degree of restricted participation are exogenous and only implicitly justified by transaction costs.}

To better motivate the notion of ‘decentralizability’, consider the example of the integration of segmented markets, e.g., the financial integration in Europe. Financial institutions and large international banks participate in all markets in different economies. In addition, a large number of the retail household investors have access only to their domestic markets, giving rise to ‘market segmentation’ or ‘restricted participation’ in several financial markets. Exchanges in different economies innovate and create new markets. The kind of questions our analysis of ‘decentralizability’ asks are: Is it possible that these different exchanges, innovating in an uncoordinated fashion, introduce a suboptimal set of innovations? Would it be economically advantageous to ‘harmonize’ by regulating the innovation process by decentralized exchanges?

We briefly report on the main results of the paper in the following. We show that the optimality of financial structures, in an economy in which lump-sum transfers compensate agents for negative relative price effects, is determined only by the ‘betas’ associated with the financial structures, i.e., by the lists of the covariances of each agent’s endowment with each asset’s payoff, normalized by the variance of the asset payoff. The optimality criterion captures in a simple expression the risk-sharing opportunities created by the financial innovation between the agents in the economy, and is represented by a particular measure of dispersion of the betas. The aggregate welfare associated with a financial market structure increases with the dispersion of the betas. For instance, an innovation consisting of the
introduction of a new asset is optimal if it maximizes the sum of the distances between the betas of each pair of agents which are allowed to trade in the new security. Similarly, an innovation consisting of the integration of two groups of agents in the same market is optimal if it maximizes the distance of the mean betas of the agents in the two groups. Consistently with our objectives, such simple characterizations of the optimality of different kinds of financial innovations are in principle amenable to an empirical analysis that estimates the welfare effects of the financial innovations over the last decades; following the lead of Davis and Willen (1999) and Athanasoulis and Shiller (2001).

We also show that optimal financial structures are ‘decentralizable’ whenever the financial innovation consists of either the introduction of new assets in an economy without restricted participation constraints, or the relaxation of restricted participation constraints for an existing asset. On the contrary, optimal financial structures may not be ‘decentralizable’ when the innovation consists of the introduction of new assets in economies with restricted participation: the introduction of new assets and the integration of segmented markets interact so that even the weak notion of optimality of financial intermediation, guaranteed by ‘decentralizability,’ is not satisfied. In presence of restricted participation, where the nature of restriction varies across financial assets, the optimality of financial market structures requires some form of coordination amongst innovating intermediaries. In this case the order in which financial innovations are introduced has potential significance in terms of welfare consideration.

The intuition for this result is simple but subtle. Each financial intermediary introduces a new asset that maximizes the welfare of its ‘participating’ agents. Trades in this new market however affect the trades of the participating agents in other markets, possibly involving some of the ‘non-participating’ agents. This in turn affects endogenously the optimality of an innovation by any other financial intermediary who maximizes the welfare of its own ‘participating’ agents, a set possibly different from the first ‘participating’ set. It is exactly the possibility of non-overlapping sets of ‘participating’ agents in different markets which potentially leads to the market failure of uncoordinated financial innovations, even if each intermediary’s objective is aligned with the welfare criterion of the ‘participating’ agents. It is precisely in these circumstances that some form of coordination of the innovating process of decentralized exchanges might prove to be efficiency enhancing.

Our theoretical results thus provide two important normative prescriptions: (i) financial innovations that generate a higher level of risk-sharing, as characterized by dispersion of betas across agents, are more desirable than others from an overall welfare standpoint; and (ii) some coordination of the innovation process is likely to be desirable when asset markets in the integrated economies are segmented with different participating agents, as, we argued, is the case for European financial integration.
For the sake of tractability, we restrict our analysis of financial innovations to two-period pure exchange Capital Asset Pricing Model (CAPM) economies in which financial markets are incomplete, and traders’ participation in financial markets is restricted. In particular, the class of economies we consider has normally distributed endowments and assets’ payoff, and agents with exponential utility. We allow agents to consume in both periods, so that agents trade to smooth wealth across time as well as to diversify risk, a real risk-free interest rate is well defined, and financial innovations affect the equilibrium risk-free rate. As a consequence, financial innovations are not necessarily Pareto improving in these economies, as they generate relative price effects which might affect negatively a subset of the agents in the economy. This allows for a much richer analysis of financial markets and innovations.

The properties of competitive equilibria for the class of economies we study have been extensively studied, notably by Willen (1997, 1999-a); see also the references therein. A characterization of optimal incomplete asset structures in one-period CAPM economies is derived in Demange and Laroque (1995). We extend their analysis (i) by considering two-period economies in which agents consume in both periods, thereby trading to smooth as well as to diversify their wealth process; (ii) by considering innovations which consist of the integration of markets as well as of the introduction of new assets; and (iii) by studying the ‘decentralizable’ of optimal financial structures and financial innovations.

Section 2 sets up the two-period CAPM economy with incomplete markets and restricted participation. Section 3 characterizes the competitive equilibrium of such an economy. Section 4 characterizes the optimality of financial markets structures, and Section 5 studies their ‘decentralizable’ by means of a sequence of financial innovations. Section 6 concludes. All the proofs not in the main body of the paper are contained in the Appendix.

2 The Economy

We study the class of simple CAPM economies introduced by Willen (1997). An economy is populated by \( H \) agents, who live for two periods, 0 and 1.

Agent \( h \in \mathcal{H} := \{1, \ldots, H\} \) has a safe endowment \( y_h^0 \) in period 0, and a random endowment \( y_h^1 \) in period 1, of the unique consumption good. Formally, \( y_h^1 \) is a random variable in a probability space \((\Omega, F, P)\).
Assumption 1  Endowments are normally distributed:

- Let $v$ denote a $N$-dimensional vector of random variables, multivariate normals with mean $0$ and variance-covariance matrix $I$, the identity matrix; endowments $y_h$, for any $h$, are

$$y_h := Y^hv, \ Y^h \in \mathbb{R}^N$$

Agent $h$ is also endowed with Von Neumann-Morgernstern preferences which associate utility index $E[u_h(c_0, c_1)]$ to any $c_0 \in \mathbb{R}$ and any random variable $c_1$ in the underlying probability space $(\Omega, F, P)$ (with respect to which the expectation operator $E[.]$ is taken).

Assumption 2  The utility function is:

1. time and state separable:

$$u^h(c_0, c_1) := u^h(c_0) + u^h(c_1(\omega)), \ \omega \in \Omega,$$

2. CARA with identical absolute risk aversion, $A > 0$, across agents:

$$u^h(c) = -\frac{1}{A}e^{-Ac}$$

In financial markets, both a risk-free bond and $J$ risky assets are traded. The bond, asset $0$ in our notation, has a payoff $x_0 = 1$ (in units of the consumption good) with probability 1. Asset $j \in J := \{1, \ldots, J\}$’s payoff, on the other hand, is a random variable in $(\Omega, F, P)$, denoted $x_j$.

Assumption 3  Assets’ payoffs are normally distributed:

- Assets payoffs $x := [x_j]_{j \in J}$ are multivariate normal random variables with mean 0 and variance-covariance matrix $I$, the identity matrix.

Trading in financial market is possibly restricted. Let $J^h$ denote the set of assets that agent $h$ is allowed to trade. Let $H_j$ denote the set of agents which are allowed to trade asset $j$.

In general, we allow for incomplete markets,

$$J < N$$

\footnote{Only notational complications are added by allowing heterogeneity in absolute risk aversion parameters.}
and restricted participation:

$$\mathcal{H}_j \subset \mathcal{H}, \text{ for some } j$$

We only impose the following assumption on the form of restricted participation, an assumption which cannot be relaxed without losing the closed form characterization of the equilibrium.

**Assumption 4** All agents $h$ are allowed to trade the risk-free bond:

$$\mathcal{H}_0 = \mathcal{H}$$

## 3 Competitive Equilibria

We study and characterize competitive equilibria for the class of two-period economies just introduced.

### 3.1 Definitions

Let $\pi_0 \in \mathbb{R}_+$ denote the price of the risk-free bond, and $\pi_j \in \mathbb{R}_j^J$ the price of the asset $j$. The problem of each agent $h$ is to choose a consumption allocation, $[c^h_0, c^h_1]$, and portfolio positions in the risk-free bond and in all tradable assets, $[\theta^h_0, \theta^h_j]_{j \in J} \in \mathbb{R}^{J+1}$, to maximize

$$u^h(c^h_0, c^h_1) := -\frac{1}{A} e^{-Ae^h} + E \left[ -\frac{1}{A} e^{-Ae^h} \right]$$

subject to the budget constraints and the restricted participation constraints:

$$c^h_0 = y^h_0 - \pi^h_0 \theta^h_0 - \sum_{j \in J} \pi_j \theta^h_j,$$  \hspace{1cm} (2)

$$c^h_1 = y^h_1 + \theta^h_0 + \sum_{j \in J} \theta^h_j x_j, \text{ and}$$

$$\theta^h_j = 0, \text{ if } j \notin J^h \hspace{1cm} (3)$$
Definition 1 A competitive equilibrium is a consumption allocation \((c^h_0, c^h_1)\), for all agents \(h\), and a price vector \(\pi := [\pi_0, \pi_j]_{j \in J}\), such that consumption and financial markets clear:

\[
\sum_h (c^h_0 - y^h_0) \leq 0, \tag{5}
\]

\[
\sum_h (c^h_1 - y^h_1) \leq 0, \text{ with probability } 1 \text{ in } (\Omega, F, P), \text{ and} \tag{6}
\]

\[
\sum_h \theta^h = 0 \tag{7}
\]

3.2 Characterization of Equilibria

Closed form solutions for equilibrium allocations and prices are easily derived (see Willen, 1997; we report the solution in the Appendix for completeness). It suffices here to note that at a competitive equilibrium,

- the price of any existing asset \(j\), \(\pi_j\), relative to the price of the bond, \(\pi_0\), is independent of the set of assets traded in the economy; i.e., it only depends on the distribution of endowments and on the properties of the distribution of asset \(j\)'s payoffs:

\[
\frac{\pi_j}{\pi_0} = E(x_j) - A \text{ cov} \left( \frac{1}{H_j} \sum_{h \in H_j} y^h_1, x_j \right), \quad j \in J \tag{8}
\]

- agents' portfolios satisfy a three-fund separation property, as each agent holds the bond, the market portfolio and the un-hedgable component of his endowment:

\[
c^h_1 = y^h_1 - \sum_{j \in J^h} \beta^h_j x_j + \sum_{j \in J^h} \beta_j x_j + \theta^h_0 \tag{9}
\]

where

\[
\beta^h_j := \frac{\text{cov} (y^h_1, x_j)}{\text{var} (x_j)}; \quad \beta_j := \frac{\text{cov} \left( \frac{1}{H_j} \sum_{h \in H_j} y^h_1, x_j \right)}{\text{var} (x_j)} \tag{10}
\]
4 Optimality

We now turn to the characterization of the optimality of financial market structures and of financial innovations.

Fixing the endowments, \([y^0_h, y^1_h]\), and the set of agents in the economy, \(H\), we parameterize a financial market structure by the list of tradable assets, their payoffs and participation sets: \([x_j, \mathcal{H}_j; j \in J]\).

To introduce general financial innovations, we say that a financial structure \(F' = [x'_j, \mathcal{H}'_j; j \in J']\) innovates on \(F = [x_j, \mathcal{H}_j; j \in J]\), if \(F'\) either adds assets or relaxes some participation restriction (or both) with respect to \(F\); i.e., if

\[ J \subseteq J', \quad x_j = x'_j, \text{ if } j \in J, \quad \text{and } \mathcal{H}_j \subseteq \mathcal{H}'_j \]

4.1 Optimal Financial Structures

As a measure of the welfare associated to an arbitrary financial market structure \(F\), we consider the average welfare gains associated to \(F\) with respect to the autarchic financial market structure, in which only the risk-free bond is traded. More precisely, the welfare of financial market structure \(F\) is measured by the \textit{compensating aggregate transfer} of \(F\), i.e., the reduction in the time 0 consumption allocation for the economy with financial structure \(F\), which, when redistributed lump sum across all agents, makes them indifferent between the allocation associated with the economy with financial structure \(F\) and the allocation associated with the economy with autarchic financial structure. The compensating aggregate transfer is the appropriate measure of the welfare gains of a particular financial structure with respect to autarchy for an economy in which lump-sum transfers across agents to redistribute welfare gains and losses can be made.\(^5\)

\(^5\)Formally, let \([c_0, c_1] := [c^0_h, c^1_h]_{h \in H}\); let \(U([c_0, c_1])\) denote the average welfare associated with the consumption allocation \([c_0, c_1]\):

\[
U([c_0, c_1]) = \frac{1}{H} \sum_{h \in H} \left( -\frac{1}{A} e^{-A c^0_h} + E \left[ -\frac{1}{A} e^{-A c^1_h} \right] \right). \tag{11}
\]

Let \([c^0_F, c^1_F]\) be the equilibrium allocation of the economy with financial structure \(F\); and let \([c^0_a, c^1_a]\) be the equilibrium allocation for the autarchic economy, in which no agent can trade any assets, except the risk-free asset. Let \(\pi^F_0\) and \(\pi^a_0\) denote the price of the risk-free asset, respectively, in the economy with financial structure \(F\) and in the autarchic economy.

The \textit{compensating aggregate transfer} of \(F\), \(\mu_F\), by definition solves

\[
U([c^0_F, c^1_F]) = U([c_0 - \mu_F, c_1]) \tag{12}
\]
Using the closed-form competitive equilibrium solution, given in Appendix, it is straightforward to show that the compensating aggregate transfer of financial market structure $F$, $\mu_F$, only depends on the equilibrium price of the risk-free bond in the economy with the financial structure $F$, $\pi_0^F$, and its counterpart in the economy with the autarchic structure, $\pi_0^a$; in particular (see Willen, 1997):

$$\mu_F = -\frac{1}{A} \ln \frac{1 + \pi_0^F}{1 + \pi_0^a} \quad (13)$$

The welfare associated to a financial market structure $F$ is decreasing with $\pi_0$, the price of the risk-free asset in equilibrium (hence is increasing with the return of the risk-free asset). The price of the risk-free asset is in fact low when the agents’ precautionary component of savings is relatively low, and hence when a large fraction of the risk in the economy is hedged.

Consider the set of economies in which, e.g., because of transaction costs, markets are not complete, $J < N$, and/or participation in financial markets is restricted, the cardinality of $H_j$ (let it be denoted by $H_j$) is $< H$, for some $j$. The degree of market incompleteness as well as the degree of restricted participation is exogenous and determines the set of feasible financial market structures for the economy. We study the optimality of financial market structures in such restricted feasible set, i.e., we provide a characterization of the financial market structure $F = [x_j, H_j; j \in J]$ which maximizes $\mu_F$, given $J$ and $H_j$, for all $j \in J$.

The next Lemma characterizes such a financial structure. We assume in what follows that $H_j \geq 2$ for any $j$, without loss of generality (assets are in zero-net-supply, and hence an asset $j$ such that $H_j = 1$ is not traded in equilibrium).

To any financial structure is associated a vector of ‘betas,’

$$\beta := [\beta_j^h], \ h \in H, \ j \in J$$

The converse is not true: assets’ payoffs $x_j$ and agents’ endowments $y^h$ are characterized by the variance of $y^h$, the variance of $x_j$, and their covariance (means are normalized to zero by Assumption 3), which cannot be recovered uniquely from knowledge of $\beta_j^h := \frac{\text{cov}(y^h, x_j)}{\text{var}(x_j)}$. Lemma 1 shows that the optimality of a financial structure can be determined by looking only at its associated vector of betas, rather then at the whole variance-covariance matrix of endowments and assets’ payoffs: the optimal financial structure maximizes a measure of the ‘dispersion’ of the betas in the population, that is the sum of the squared distances between the betas of each pair of agents in each possible market.
Lemma 1 (Beta Representation) The compensating aggregate transfer $\mu_F$ is maximal for the financial structure $F$ whose betas maximize

$$\sum_{j \in J} \sum_{h \in H_j} \sum_{h' \in H_j} \frac{1}{H_j} \left( \beta_{hj} - \beta_{h'j} \right)^2$$

(14)

Such characterization can be substantially sharpened if we consider economies in which each agent’s participation in financial markets is unrestricted, i.e., $H_j = H$, for any $j$.

Suppose the economy’s financial market structure is restricted to be composed of $J$ assets which all agents can trade; which are the $J$ optimal assets for such an economy, i.e., the $J$ assets whose betas maximize (14)? The next proposition answers this question.

Proposition 1 (Principal Components Characterization) Suppose that market participation is not restricted in any asset:

$$H_j = H, \ \forall j \in J.$$

Then a financial structure $F$ maximizes the compensating aggregate transfer $\mu_F$ if the asset payoffs, $[x_j]$, are a linear combination of the agents’ endowments:

$$x_j := R_j v, \ \ \ R_j \in \mathbb{R}^N;$$

and the columns of $R' := [R_j]'$ are spanned by the $J$ principal components associated with the $J$ largest eigenvalues of the matrix

$$M = \sum_{h \in H} (Y^h - Y)'(Y^h - Y),$$

where $Y = \frac{1}{\#H} \sum_{h \in H} Y^h$.

If the participation in any asset is unrestricted, the optimal financial structure of $J$ assets is composed of the assets whose payoffs are those linear combinations of endowments that produce a maximum dispersion of betas across the agents. In particular, the optimal assets are the eigenvectors corresponding to the $J$-th largest eigenvalues of the matrix $M$ above. The matrix, $M$, captures the extent of risk-sharing provided by the assets in the financial structure $J$.\footnote{This characterization of optimal financial structures coincides with the one derived by Demange and Laroque (1995) for their one-period economy. This is perhaps surprising, since our economy differs in an important from Demange-Laroque’s in that we allow for consumption in period 0, and hence we allow for welfare effects through changes in the risk-free rate at the competitive equilibrium.} In fact, it represents the dispersion of the agents’ endowments along the bases
of the endowment space. Thus, analogously to factor analysis, the optimal assets are the ‘factors’ that capture as much of this dispersion as possible, given the restriction on the number of factors. The proposition above demonstrates that the optimal assets are indeed the principal components of the dispersion matrix, $M$, which explain the highest possible dispersion, i.e., which correspond to the largest eigenvalues.

The characterization of optimal financial market structures in Lemma 1 can also be specialized to study the optimal composition of the agents trading a given asset $j$, in terms of their betas. Given an arbitrary asset $j$, with payoff $x_j$, and given a maximal number of agents who are allowed to participate in trading asset $j$, $H_j$, what is the optimal composition of the set of agents who are allowed to trade, i.e., the set $\mathcal{H}_j$ such that (14) is maximized? The next proposition answers this question.

**Proposition 2 (Traders’ Composition)** Assume that agents’ betas for asset $j$ are distinct, and order agents so that

$$\beta^N > \cdots > \beta^h > \cdots > \beta^1.$$

There exists an $r < H_j$ such that the optimal composition of agents trading asset $j$ consists of the first $r$ agents and the last $H_j - r$ agents; i.e., $\mathcal{H}_j = \{1, \ldots, r, H - (H_j - r) + 1, \ldots, H\}$.\footnote{See Lemma A.1 in the Appendix, where the Proposition is proved.}

The optimal composition of agents allowed to trade an arbitrary asset $j$ in an economy with restricted participation consists of two sets of agents, the sets being at the two extremes of the ranked betas of the agents with respect to the asset. The distribution of agents in the two extreme sets depends, in general, on the structure of the endowments. Once again, this characterization has the general flavor of our results that optimal financial market structures, defined as per our welfare criterion, maximize a measure of the dispersion of betas across the agents affected by the innovations.

### 4.2 The Welfare Effects of Financial Innovations

It follows from (13) that the measure of the welfare gains associated to a financial structure $F'$ which innovates on $F$, with respect to $F$, is $\mu_{F'} - \mu_F$. The following is then a simple implication of Lemma 1.

**Proposition 3** If $F'$ innovates on $F$, no welfare losses are possible, $\mu_{F'} \geq \mu_F$. Moreover, strictly positive welfare gains are realized, $\mu_{F'} > \mu_F$, if the innovation involves the integration
of agents with different betas, or the introduction of an asset tradable by agents with different betas, i.e., if for some $j \in \mathcal{J}'$, $h, h' \in \mathcal{H}_j$ such that either $j \notin \mathcal{J}$ or $h' \notin \mathcal{H}_j$, $\beta^{h_j} \neq \beta^{h'_j}$.

Thus, financial innovations invariably have positive welfare effects, as measured by the aggregate compensation transfer. However, a subset of the agents might have to be compensated by lump-sum transfers after a financial innovation, as they might experience a loss due to the change in the risk-free asset’s price which is a consequence of the innovation. This is a feature of the model with consumption in period 0; the one-period CAPM economy studied by Demange and Laroque (1995) has no price effects due to financial innovations.

We study separately the welfare effects of two standard innovations: (i) the introduction of a new financial asset, and (ii) the integration of two distinct markets for the same financial asset. We consider first financial innovations consisting of the introduction of an asset, given a set of pre-existing assets in the economy.

**Proposition 4 (Introduction of a new asset)** Suppose a financial innovation consists of introducing a new asset $j'$, i.e., $F'$ is as $F$ except that $\mathcal{J}' = \mathcal{J} \cup \{j'\}$. The welfare gain of such innovation, $\mu_{F'} - \mu_F$, is increasing in

$$\sum_{h \in \mathcal{H}_{j'}} \frac{1}{H_{j'}} \left( \frac{\beta^{h}_{j'} - \beta^{h'}_{j'}}{1} \right)^2$$

(15)

The welfare gains due to the introduction of a new asset $j'$ depend on the ‘dispersion,’ across the agents allowed to trade the asset, of the betas relative only to that same asset. In other words, the welfare gains associated to the introduction of asset $j'$ are independent of the betas relative to all assets traded before the innovation. An innovation consisting of the introduction of a single asset with unlimited participation is optimal if its ‘betas’ are maximally dispersed across the agents. The dispersion of the betas relative to the new asset, measured by the average of the squared deviation of the betas, captures the extent of risk-sharing introduced by the asset.9

We can now turn to study the welfare effects of innovations which consist of the integration of two distinct markets for the same financial asset.

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8Willen 1999-b addresses the difficult question of the distributional effects of financial innovations across heterogeneous agents.

9In a related set-up, but with no restricted participation, Duffie and Jackson (1989) show that the optimal innovation in the financial asset market maximizes the total transaction volume. It can be shown (Lemma 2, Appendix 2) that our optimality criterion is equivalent to maximizing the sum of squared transaction volume. This difference in the characterization of the optimal asset arises due to a difference in the notions of optimality employed, as Duffie and Jackson study the Pareto optimality of the innovation without allowing for lump-sum transfers.
Proposition 5 (Integration of two distinct markets) Suppose the financial structure 
$F$ has the property that assets $j$ and $j'$ have the same payoff, $x_j = x_{j'}$, but are traded in 
distinct markets, $H_j \cap H_{j'} = \emptyset$. A financial innovation which integrates such markets, i.e., a 
financial structure $F'$ which is as $F$ except that $H'_{j} = H_j \cup H_{j'}$, has a welfare gain $\mu_{F'} - \mu_F$ 
which is increasing in

$$
\frac{H_j H_{j'}}{H_j + H_{j'}} (\beta_j - \beta_{j'})^2
$$

(16)

The welfare gains of innovations which consist of the integration of markets increase in 
the number of agents integrated. Most importantly, keeping constant the number of agents 
in each group, the welfare gains of market integration only depend on the difference between 
the average betas of the two groups, $\beta_j$ and $\beta_{j'}$, and do not depend on the individual betas 
of the agents in the groups. Keeping constant the size of the markets, the integration of 
two distinct markets is optimal when no other pair of markets exists whose difference in 
average betas is higher. For example, the welfare gain of allowing agent $h'$ to trade asset 
$j$ is increasing in $(\beta_j - \beta_{h'})^2$, the difference between the beta of agent $h'$ with respect to 
asset $j$ and the average of the betas of agents trading asset $j$ before the integration; and the 
integration in some market $j$ of agents whose beta is right at the average of the betas of the 
traders in that market has no welfare effects.

5 Financial Innovations and Decentralizability

For an economy in which, e.g., because of transaction costs, markets are not complete, 
$J < N$, and/or participation in financial markets is restricted, $H_j \subset H$ for some $j$, we 
have provided in the previous section a characterization of the optimal financial market 
structure $F = [x_j, H_j; j \in J]$. But can such financial market structure be decentralized 
as an equilibrium of economies in which financial intermediaries interact to introduce and 
intermediate new securities and to integrate segmented markets?

We do not model in this paper the process by which innovations are determined in fi-
nancial markets. The degree of market incompleteness as well as the degree of restricted 
participation are exogenous and only implicitly justified by transaction costs, and hence 
we cannot attempt to answer this question directly. Nevertheless our analysis potentially 
sheds light on the ‘decentralizability’ of optimal financial asset structure by means of financial 
innovations introduced sequentially by different intermediaries. More precisely, suppose 
that, given $J$ and $H_j$ for any $j \in J$, each innovation is introduced in financial markets in-
dependently of the others, so as to satisfy the orthogonality of asset payoffs, and so as to
maximize the optimality criterion, (14), for given existing financial market structure. Innovations might consist either of the introduction of a new asset or of the integration of two markets, or both. Does the financial market structure which results from such a sequence of financial innovations necessarily coincide with the optimal financial market structure? If this is so, we say that the optimal financial asset structure is 

\textit{decentralizable}.

It is important to note that ‘decentralizability,’ as defined in this paper, is far from sufficient for financial intermediation to be optimal. In general, allowing for assets with correlated payoffs, and for innovations to maximize e.g., trading volume (as in Duffie and Jackson, 1989), or intermediation profits (as in Pesendorfer, 1995), rather than an optimality criterion, might in fact introduce inefficiencies in the design of assets. By concentrating on ‘decentralizability’ we abstract from such sources of inefficiency and concentrate rather on the possible inefficiency resulting from the uncoordinated introduction of innovations in the markets for financial intermediation.

\section{5.1 Decentralizability}

We turn now to the ‘decentralizability’ of the optimal financial market structure by means of independent financial innovations. A precise definition of ‘decentralizability,’ which applies generally to optimal asset structures as well as to optimal compositions of traders, is as follows.

\textbf{Definition 2} Let the optimal financial structure of an economy with $J$ orthogonal financial assets and market participation structure $\mathcal{H}_j$, $j = 1, \ldots, J$ be denoted $F$. The betas associated with $F$ maximize $\sum_{j \in J} \sum_{h \in \mathcal{H}_j} \sum_{h' \in \mathcal{H}_j} \frac{1}{I_j}(\beta^h_j - \beta^{h'}_j)^2$ (by Lemma 1). Consider then another financial structure, $\hat{F}$, with $J$ orthogonal financial assets and market participation structure $\mathcal{H}_j$, $j = 1, \ldots, J$. Suppose $\hat{F}$ satisfies the following:

- the betas associated with each asset $j$ maximize $\sum_{h \in \mathcal{H}_j} \sum_{h' \in \mathcal{H}_j} \frac{1}{I_j}(\beta^h_j - \beta^{h'}_j)^2$, given $\mathcal{H}_j$;

- for any asset $j$, the components of $\mathcal{H}_j$ can be ordered so that, without loss of generality,

$$h := \arg\max_{h' \geq h} \sum_{h'' = 1}^{h-1} (\beta^{h''}_j - \beta^{h'}_j)^2.$$

We say that the optimal financial structure $F$ is ‘decentralizable’ if $F$ does not strictly dominate $\hat{F}$, in welfare terms, i.e. $\mu_F = \mu_{\hat{F}}$.\footnote{Since $F$ is optimal, obviously, $\mu_F \geq \mu_{\hat{F}}$.}
The rationale behind studying ‘decentralizability’ is to see if a set of innovations that is introduced either sequentially or in an uncoordinated manner by a group of financial intermediaries leads to a financial structure that is overall optimal. If ‘decentralizability’ fails to hold, it suggests that there are costs from having decentralized exchanges or intermediaries introducing innovations independently (in an uncoordinated fashion). The efficiency gains (from a financial optimality standpoint) that result from a harmonization of the innovation process may thus be significant for market structures that lack ‘decentralizability.’

In the case where financial innovations are restricted to the introduction of new assets, for instance, ‘decentralizability’ requires that the optimal financial structure can be obtained equivalently by introducing a first financial asset \( x_1 \) which maximizes

\[
\sum_{h \in \mathcal{H}_1} \sum_{h' \in \mathcal{H}_1} \frac{1}{H_1} \left( \beta_h^1 - \beta_{h'}^1 \right)^2;
\]

a second asset whose payoff is orthogonal to \( x_1 \) above, and maximizes

\[
\sum_{h \in \mathcal{H}_2} \sum_{h' \in \mathcal{H}_2} \frac{1}{H_2} \left( \beta_h^2 - \beta_{h'}^2 \right)^2;
\]

and so on sequentially until the \( J \)-th asset. Similarly, for the case of market integration in a given asset, ‘decentralizability’ requires that a sequential strategy of adding an optimal agent, given the existing participation in the asset, gives rise to the overall optimal composition of traders in that asset.

We examine the ‘decentralizability’ of the optimal financial market structures for economies (i) in which the participation of the agents in all financial markets is unrestricted, but the number of tradable assets is limited; (ii) in which the number and the payoffs of tradable assets are exogenously determined, but the number of agents allowed to trade each particular asset is limited; and, finally, economies (ii) in which the agents’ participation in financial markets is exogenously determined, and the number of tradable assets is limited.

Consider first an economy in which the optimal financial structure consists of \( J \) assets with no restriction in participation. The following is a simple implication of the principal components characterization (Proposition 1).

**Proposition 6** The financial structure \( F \), which is optimal in the class of financial structures with \( J \) assets and such that participation is not restricted in any asset, \( \mathcal{H}_j = \mathcal{H}, \ \forall j \in \mathcal{J} \), is ‘decentralizable.’

The optimal financial asset structure defined by the principal component characterization has the property that the \( n \)-th asset is chosen so that the asset’s payoff equals the eigenvector
corresponding to the $n$–th largest eigenvalue of the matrix $M$ defined in Proposition 3. The $n$–th asset in a sequence of financial innovations is chosen in exactly the same way.

We next examine the ‘decentralizability’ of the optimal financial asset structure of an economy in which the payoff of each asset available for trade is exogenously determined, and the number of agents allowed to trade each particular asset is limited.

**Proposition 7** The financial structure $F$, which is optimal in the class of financial structures with $J$ assets with exogenously given payoff, and such that asset $j$ is traded by no more than $n < H$ agents, is ‘decentralizable.’

The result can be extended to the case in which the restriction on the participation in financial market $j$ depends on $j$ itself, i.e., in which no more that $H_j$ agents are allowed to trade asset $j$, with $H_j < H$ for some $j$.

It is then interesting to study if ‘decentralizability’ holds when assets are introduced in an economy in which the structure of the restriction in the market participation is given exogenously. The ‘decentralizability’ of optimal financial structures associated with the principal components characterization (Proposition 6) in fact relies crucially on the assumption that the market participation structure is assumed to be the same across all the assets. The next proposition shows that, if the restricted market participation structure is different across different assets, the sequential introduction of financial assets can produce a financial market structure that is not optimal amongst all structures with the same number of financial assets.

In other words, with restricted participation, the order of innovation of financial asset may affect aggregate welfare.

**Proposition 8** The financial structure $F$, which is optimal in the class of financial structures with $J$ assets and given restricted participation structure $\mathcal{H}_j$, with $\mathcal{H}_j \subset \mathcal{H}$ for some $j$, may not be ‘decentralizable.’

**Proof of Proposition 8.** We will prove the proposition by introducing an example economy in which the optimal financial structure can, in fact, strictly dominate the financial structure resulting by the sequential introduction of optimal innovations.

Consider an economy with $H = 3$ i.e. $\mathcal{H} = \{0, 1, 2\}$. Also the dimension of endowment space is $N = 3$. The agents’ endowments are given by:

$$Y_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad Y_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \quad Y_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

The matrix $M$ in the principal components characterization is
\[ M = \frac{1}{3} \begin{bmatrix} 2 & -1 & -2 \\ -1 & 6 & 4 \\ -2 & 4 & 8 \end{bmatrix}, \]

whose eigenvalues are \( \lambda_1 = 11.62 \), \( \lambda_2 = 3.00 \), and \( \lambda_3 = 1.38 \), with the corresponding eigenvectors:

\[
x_1 = \begin{bmatrix} -0.27 \\ -0.80 \\ 0.53 \end{bmatrix}; \quad x_2 = \begin{bmatrix} 0.22 \\ -0.59 \\ -0.78 \end{bmatrix}; \quad x_3 = \begin{bmatrix} -0.94 \\ 0.09 \\ -0.34 \end{bmatrix}.
\]

Thus, in absence of any restricted market participation, for \( J = 1 \), the optimal financial structure is \((x_1)\), and for \( J = 2 \), the overall optimal financial structure is \((x_1, x_2)\).

Next consider economies with \( J = 2 \) financial assets and with an exogenously given restricted market participation structure of \( \mathcal{H}_1 = \{0, 1\} \), and \( \mathcal{H}_2 = \{1, 2\} \). The solution to the optimization problem for the overall financial structure yields the optimal assets as:

\[
x_1 = \begin{bmatrix} 0.10 \\ -0.28 \\ -0.96 \end{bmatrix}; \quad x_2 = \begin{bmatrix} 0.48 \\ 0.86 \\ -0.20 \end{bmatrix}.
\]

The corresponding betas of the participating agents are \( \beta^0_1 = -3.32 \), \( \beta^1_1 = -1.13 \), and \( \beta^1_2 = 1.14 \), \( \beta^2_2 = 3.33 \). The welfare measure for this structure is

\[
\ln(\mu_F) = k_1 + k_2 \cdot \sum_{j=1}^{2} \sum_{h \in \mathcal{H}_j} \sum_{h' \in \mathcal{H}_j} \frac{1}{2} (\beta^h_j - \beta^{h'}_j)^2 \\
= k_1 + k_2 \cdot \frac{1}{2} (4.791 + 4.791) = k_1 + k_2 \cdot 4.791,
\]

where \( k_1 \) and \( k_2 \) are positive constants irrelevant to the analysis.

On the other hand, for the case of sequentially optimal asset introduction, solutions to the optimization problems are given as:

\[
x_1 = \begin{bmatrix} 0.00 \\ -0.45 \\ -0.89 \end{bmatrix}; \quad x_2 = \begin{bmatrix} 0.49 \\ 0.78 \\ -0.39 \end{bmatrix}.
\]

The corresponding betas of the participating agents are given as: \( \beta^0_1 = -3.58 \), \( \beta^1_1 = -1.34 \), and \( \beta^1_2 = 0.88 \), \( \beta^2_2 = 2.93 \). Thus, the welfare measure can be computed as

\[
\ln(\mu_F) = k_1 + k_2 \cdot \sum_{j=1}^{2} \sum_{h \in \mathcal{H}_j} \sum_{h' \in \mathcal{H}_j} \frac{1}{2} (\beta^h_j - \beta^{h'}_j)^2 \\
= k_1 + k_2 \cdot \frac{1}{2} (5.000 + 4.1999) = k_1 + k_2 \cdot 4.599,
\]
Note that the welfare measure is smaller than that with overall optimal structure ($k_1$ and $k_2$ are the same for a given economy). ♦

This example illustrates the fact that a sequentially optimal structure may not do as well in welfare terms as an overall optimal structure in the presence of restricted market participation; and this is because of lack of coordination in the innovation process. As an intuition of the result, notice that the first sequential asset produces a welfare change greater than each of the two assets under the overall optimal structure. However, the first sequential asset does not take into account the restricted market participation of the second asset (which is different than that of the first asset), and as a result, the second sequential optimal asset is inferior in welfare terms than both the overall optimal assets. In fact, it is inferior enough that the sequential structure is strictly dominated by the overall structure.

We have already shown in Proposition 6 and Proposition 7 that optimal financial structures are 'decentralizable' whenever the financial innovation consists of either the introduction of new assets in an economy without restricted participation constraints, or the relaxation of restricted participation constraints for an existing asset. Proposition 8 shows that on the contrary, optimal financial structures are not 'decentralizable' when the innovation consists of the introduction of new assets in economies with restricted participation: the introduction of new assets and the integration of segmented markets interact so that even the weak notion of optimality of financial intermediation, guaranteed by 'decentralizability,' is not satisfied. In presence of restricted participation, where the nature of restriction varies across financial assets, the optimality of financial market structures requires some form of coordination amongst innovating intermediaries. In this case the order in which financial innovations are introduced has potential significance in terms of welfare consideration.

It is exactly the different forms of segmentation and 'participation' in different markets which might lead to the market failure of uncoordinated financial innovations. The creation of individual markets which are optimal given the existing market structure might aggregate in this case into an overall financial structure that is less than optimal. Each financial intermediary introducing a new asset that maximizes the welfare of its 'participating' agents in fact affects the trades of the participating agents in other markets, possibly involving some of the agents which are restricted out of its own market. This in turn affects endogenously the innovation by any other financial intermediary who maximizes the welfare of its own 'participating' agents. It is precisely in these circumstances then that some form of coordination of the innovating process of decentralized exchanges might prove to be efficiency enhancing.

Further, this intuition suggests that the construction in the example above is in fact quite robust. A careful inspection of the endowments of different agents and the market participation structures reveals the following. If introducing an asset generates a large amount
of risk-sharing between its participating agents (agent 0 and agent 1 for the first asset), such that it restricts the risk-sharing between some of the participating agents and the non-participating agents (agent 1 and agent 2, respectively), then the asset introductions that follow in the sequentially optimal design are likely to produce a less-than-optimal overall financial structure. The intuition being compelling enough, we do not delve into a more rigorous analysis of precise financial structures where ‘decentralizability’ fails to hold.

This result on the lack of decentralizability has implications for recent examples of the integration of segmented markets, most notably for the financial integration in Europe. Financial institutions and large international banks participate in all markets in different economies of European Union. However, a large number of the retail household investors have access only to their domestic markets, giving rise to ‘market segmentation’ or ‘restricted participation’ in several financial markets. Since exchanges in different economies innovate and create new markets, our analysis suggests that some coordination of the innovation process is likely to be desirable.

We conjecture that a similar result, the lack of ‘decentralizability,’ holds for financial market structures which are optimal in the class of financial structures in which only the number of tradable assets and the number of the agents allowed to trade each asset are given.

6 Conclusions

The theoretical results of this paper provide two important normative prescriptions: (i) financial innovations that generate a higher level of risk-sharing, as characterized by dispersion of betas across agents, are more desirable than others from an overall welfare standpoint; and (ii) some form of harmonization or coordination of the innovations process of decentralized financial intermediaries is likely to be desirable when asset markets in the integrated economies are segmented with different participating agents.

While this paper only analyzes the optimality of financial structures for economies in which lump sum transfers across agents are possible, many interesting issues arise with regards to the individual welfare effects of financial innovations, i.e., to the welfare effects when lump sum transfers are not implementable. In this case in fact, the welfare effect of financial innovations can in general be negative for a subset of the agents, as the relative price effects of innovation have welfare effects: an innovation leads to a lower demand for precautionary savings, and hence for the riskless asset. The riskless rate of interest in the economy as a consequence increases, and the agents who need to borrow in equilibrium and do not benefit much from risk-sharing provided by financial innovation, are relatively hurt.

Unfortunately few implications of optimal financial innovations with respect to individual
welfare can be derived analytically. Some results are contained though in Willen (1999-b).

Finally, because of the simple structure of competitive equilibrium allocations of CAPM economies, our analysis of optimal financial structure can in principle be extended to CAPM economies with infinite horizon. Many new conceptual and technical issues arise however in this case, which we leave for future work.
APPENDIX 1: Competitive equilibrium (Willen, 1997)

The competitive equilibrium of the two-period CAPM economy, defined by equations (1)-(7), is characterized by prices of assets \( \pi_j \), portfolio choices \( \theta_j \), and consumption allocations \( c_t^h \), given below. Note that \( j = 0 \) denotes the riskfree asset.

\[
\pi_0 = \exp \left\{ A (y_0 - E y_1) + \frac{A^2}{2H} \sum_{h \in H} \left[ (1 - R^2_h) \text{var}(y_1^h) + \sum_{j \in J^h} \text{var}(\beta_j x_j) \right] \right\} \quad (17)
\]

where

\[
R^2_h := \frac{\sum_{j \in J^h} (\beta_j^h)^2 \text{var}(x_j)}{\text{var}(y_1^h)} \quad (18)
\]

\[
\frac{\pi_j}{\pi_0} = E(x_j) - A \text{cov} \left( \frac{1}{H^j} \sum_{h \in H^j} y_1^h, x_j \right), \quad j \in J
\]

\[
\theta_j^h = \beta_j - \beta_j^h, \quad j \in J^h, \quad \text{and} \quad \theta_0^h = 0, \quad j \in (J^h)^c \quad (19)
\]

\[
\theta_0^h = \frac{1}{1 + \pi_0} \left( y_0^h - E(y_1^h) - \sum_{j \in J^h} \pi_j^h + \frac{A}{2} \text{var}(c^1) - \frac{1}{A} \ln(\pi_0) \right) \quad (20)
\]

\[
c_t^h = y_1^h - \sum_{j \in J^h} \beta_j^h x_j + \sum_{j \in J^h} \beta_j x_j + \theta_0^h \quad (21)
\]

\[
\text{var}(c_t^h) = \text{var}(y_1^h) - \sum_{j \in J^h} (\beta_j^h)^2 + \sum_{j \in J^h} \beta_j^2 \quad (22)
\]

Consumption allocations \( c_0^h \) can be solved for using (17-22) and the budget constraint (2):

\[
c_0^h = - \frac{1}{A} \ln \frac{1}{\pi_0} + E(y_1^h) + \theta_0^h - \frac{A}{2} \text{var}(c_t^h) \quad (23)
\]
APPENDIX 2: Proofs

Proof of Lemma 1. From Willen (1997),

$$\mu_F = -\frac{1}{A} \ln \frac{1 + \pi_0}{1 + \pi_0^F} = -\frac{1}{A} \ln \frac{1 + \pi_0}{1 + \frac{\pi_0}{\pi_0^F}}$$

where for simplicity, we denote $\pi_0^F$ as simply $\pi_0$. Thus, aggregate welfare is maximized when the ratio, $\frac{\pi_0}{\pi_0^F}$, is minimized. But,

$$\pi_0 = \exp \left\{ A (y_0 - E y_1) + \frac{A^2}{2H} \sum_{h \in H} \text{var}(y_h^1) \right\},$$

and

$$\pi_0^F = \exp \left\{ A (y_0 - E y_1) + \frac{A^2}{2H} \sum_{h \in H} \left( (1 - R_h^2) \text{var}(y_h^1) + \sum_{j \in J_h} \text{var}(\beta_j x_j) \right) \right\}$$

Using $R_h^2 \text{var}(y_h^1) = \sum_{j \in J_h} (\beta_j^h)^2 \text{var}(y_h^1)$,

$$\frac{\pi_0}{\pi_0^F} = \exp \left\{ \frac{A^2}{2H} \sum_{h \in H} \left( - \sum_{j \in J_h} (\beta_j^h)^2 + \sum_{j \in J_h} (\beta_j)^2 \right) \text{var}(x_j) \right\}$$

which, in turn, can be written as:

$$\frac{\pi_0}{\pi_0^F} = \exp \left\{ \frac{A^2}{2H} \sum_{j \in J} \sum_{h \in H_j} (\beta_j^2 - (\beta_j^h)^2) \text{var}(x_j) \right\}$$

(25)

Then, using $\beta_j = \frac{1}{H_j} \sum_{h \in H_j} \beta_j^h$, we get

$$\sum_{h \in H_j} (\beta_j^2 - (\beta_j^h)^2) = \frac{1}{H_j} \left( \sum_{h \in H_j} \beta_j^2 \right)^2 - \sum_{h \in H_j} (\beta_j^h)^2 = -\frac{1}{H_j} \sum_{h \in H_j} \sum_{h' \in H_j} \left( \beta_j^h - \beta_j^{h'} \right)^2,$$

and hence, given $\text{var}(x_j) \equiv 1$, we have the result

$$\frac{\pi_0}{\pi_0^F} = \exp \left\{ \frac{A^2}{2H} \sum_{j \in J} \sum_{h \in H_j} \sum_{h' \in H_j} -\frac{1}{H_j} \left( \beta_j^h - \beta_j^{h'} \right)^2 \right\} \diamond$$

(26)
Proof of Proposition 1. We first derive another characterization of the optimality of financial structures in terms of portfolio volumes.

**Lemma 2 (Portfolio Representation)** The compensating aggregate transfer $\mu_F$ is maximal for the financial structure $F$ whose equilibrium trading portfolios, $[\theta^h_{j}]_{j \in J}$ maximize

$$\sum_{j \in J} \sum_{h \in H_j} \sum_{h' \in H_j} \frac{1}{H_j} \left( \theta^h_j - \theta^{h'}_j \right)^2.$$  \hspace{1cm} (27)

This is equivalent to maximizing

$$\sum_{j \in J} \sum_{h \in H_j} \frac{1}{H_j} \left( \theta^h_j \right)^2.$$ \hspace{1cm} (28)

**Proof of Lemma 2.** Follows from Lemma 1, equation (26), and $\theta^h_j = \beta_j - \beta^h_j$, for $j \in J^h$. The second representation is a result of the fact that $\sum_{h \in H_j} \theta^h_j = 0$. ◊

Assume $H_j = H$, for any $j \in J$. In an abuse of notation, we shall use $\mu_F$ to represent the measure of beta dispersion that maximizes it (See Beta Representation of Lemma 1). This is of course innocuous as far as the proof below is concerned. From Lemma 2, using $\theta^h_j = \beta_j - \beta^h_j$,

$$\mu_{F'} - \mu_F = \sum_{j \in J} \sum_{h \in H_j} \sum_{h' \in H_j} \frac{1}{H_j} (\beta^h_j - \beta^{h'}_j)^2$$

But, using the assumption $H_j = H$,

$$\mu_{F'} - \mu_F = \sum_{h \in H} \sum_{j \in J} (\beta^h_j - \beta^{h'}_j)^2 = \sum_{h \in H} \sum_{j \in J} \left[ \text{cov}(y^h_1, x_j) - \text{cov}(y_1, x_j) \right]^2$$

Let $y^h_1 = Y^h v$, and $Y = \frac{1}{H} \sum_{j \in J} Y^h$; i.e., $y_1 = Y v$. Since assets’ payoffs are normally distributed (Assumption 3), we can write $x = R'v + \epsilon$, with $\text{cov}(\epsilon, v) = 0$. We will first derive the result for $\epsilon = 0$, i.e., assets’ payoffs are linear in the agents’ endowments; and we will then show that such linearity actually must hold if the asset structure is optimal.

Let $R = \begin{bmatrix} R_1 \\ \vdots \\ R_J \end{bmatrix}$, and let $R^T$ denote the transpose of $R$. Then

$$\mu_{F'} - \mu_F = \sum_{h \in H} \sum_{j \in J} \text{cov}(y^h_1 - y_1, x_j)^2 = \sum_{h \in H} \sum_{j \in J} \text{cov}(Y^h v - Y v, R_j v)^2$$
$= \sum_{h \in H} \sum_{j \in J} (Y^h - Y) R_j^T R_j (Y^h - Y)^T$

Since $R_j^T R_j' = 0$, for any $j \neq j'$, and $R_j^T R_j = 1, \forall j$, it follows that $R R^T$ is an identity matrix. Then

$\mu_{F'} - \mu_F = \sum_{h \in H} \sum_{j \in J} (Y^h - Y) R_j^T R_j R_j^T R_j (Y^h - Y)^T$

$= \sum_{h \in H} (Y^h - Y) R R (Y^h - Y)^T.$

The rest of the proof for the case $x = R'v$ follows Proposition 2.3 of Demange and Laroque (1995), Page 226.

But in general, assets’ payoff have the form $x = R'v + \epsilon$. It remains to be shown that $\epsilon = 0$ is required by optimality. If $x = R'v + \epsilon$, $\beta^h = \frac{\text{cov}(R'v + \epsilon, Y^h)}{\text{var}(x)} = \frac{R^h v^h}{R v^2}$ (since $\text{var}(v) = 1$). Since the welfare criterion, (15), is homogeneous of degree 2 in betas, the optimal asset structure has $\epsilon = 0$. ♦

Proposition 2 is implied by Lemma A.1, stated and proved in Proposition 7 below.

**Proof of Proposition 3.** From $\mu_F = -\frac{1}{A} \ln \frac{1 + \pi_0}{1 + \pi_0'}$,

$\mu_{F'} - \mu_F = -\frac{1}{A} \ln \frac{1 + \pi_0'}{1 + \pi_0} = -\frac{1}{A} \ln \frac{1 + \frac{\pi_0'}{\pi_0}}{1 + \frac{1}{\pi_0}}$

Thus, as in Lemma 1, aggregate welfare is maximized when the ratio $\frac{\pi_0'}{\pi_0}$ is minimized. Fix $H_j$, for any $j \in J$. Then from Lemma 1,

$\frac{\pi_0'}{\pi_0} = \exp\left\{ \frac{A^2}{2H} \sum_{j \in J} \sum_{h \in H_j} \sum_{h' \in H_j} -\frac{1}{H_j} \left( \beta^h_j - \beta^{h'}_j \right)^2 \right\}$

It follows that $\mu_{F'} > \mu_F$ if the innovation consists of introducing an asset $j' \notin J$, such that there exist $h, h' \in H_j'$ for which $\beta^h_j = \beta^{h'}_j$.

To study the case of market integration, suppose that there exist $j, j' \in J$ such that $x_j \equiv x_{j'}$ and $H_j \cap H_{j'} = \emptyset$, and consider a new economy in which $H_j' = H_{j'} = H_j \cup H_{j'}$. Then

$\frac{\pi_0'}{\pi_0} = \exp\left\{ \frac{A^2}{2H} \sum_{h \in H_j} \sum_{h' \in H_{j'}} \left( \frac{(\beta^h_j - \beta^{h'}_j)^2}{H_j + H_{j'}} - \frac{(\beta^h_j - \beta^{h'}_j)^2}{H_j} - \frac{(\beta^h_j - \beta^{h'}_j)^2}{H_{j'}} \right) \right\}$
Using (25):

\[
\frac{\pi'_0}{\pi_0} = \exp \left\{ \frac{A^2}{2H} \left( \sum_{h \in H'_j} (\beta'_j)^2 - (\beta'_j)^2 - \sum_{h \in H_j} (\beta_j)^2 - (\beta'_j)^2 - \sum_{h \in H'_{j'}} (\beta_{j'})^2 - (\beta_{j'})^2 \right) \right\}
\]

equals

\[
\exp \left\{ \frac{A^2}{2H} \left( \sum_{h \in H'_j} (\beta'_j)^2 - \sum_{h \in H_j} (\beta_j)^2 - \sum_{h \in H'_{j'}} (\beta_{j'})^2 \right) \right\}
\]

But since, \( \beta'_j = \frac{H_j \beta_j + H_{j'} \beta_{j'}}{H_j + H_{j'}} \), we have

\[
\frac{\pi'_0}{\pi_0} = \exp \left\{ \frac{A^2}{2H} \cdot \frac{-H_j H_{j'}}{H_j + H_{j'}} (\beta_j - \beta_{j'})^2 \right\} \quad (29)
\]

It follows that, if the innovation consists of integrating asset \( j \) and \( j' \), \( \mu_{F'} > \mu_F \) if \( \beta_j \neq \beta_{j'} \).

Necessary condition for \( \beta_j \neq \beta_{j'} \) is that there exist \( h \in H_j, \ h' \in H_{j'} \) such that \( \beta_{j'}^h \neq \beta_{j'}^{h'} \) (but \( \beta_{j'}^h = \beta_{j'}^{h'} \), since \( x_j = x_{j'} \)).

Proposition 4 and Proposition 5 follow directly from Proposition 3. Proposition 6 follows immediately from Proposition 1.

**Proof of Proposition 7.** In the notation which follows the index referring to an arbitrary security \( j \) is dropped for simplicity.

Let \( H \) denote the number of agents in the population. Let \( n \) denote the maximum number of agents allowed to trade an arbitrary security. We study the optimal choice of the \( n \) agents. The only interesting case to consider is the case \( 2 \leq n < H \), obviously. Also, we restrict ourselves to the case in which agents’ betas (for the arbitrary asset) are distinct. (The proof can be extended to take into account of identical betas with only notational complications.)

This allows us to order agents, without loss of generality, so that

\[
\beta^n > \ldots > \beta^h > \ldots > \beta^1.
\]

Two definitions of optimality are compared, which we call ‘optimality’ and ‘sequential optimality’ in the following. Let \( O^n \subset H \) denote the optimal set of agents, while \( S^n \subset H \) denote the sequential optimal set of agents (both sets have cardinality \( n \) by construction, of course).
In particular, if \( H^n \) denotes any arbitrary \( n \)-dimensional subset of \( H \), and

\[
\overline{\beta}^n = \frac{1}{n} \sum_{h \in H^n} \beta^h,
\]

\[
O^n := \arg\max_{H^n} \sum_{h \in H^n} (\beta^h)^2 - n \left( \overline{\beta}^n \right)^2.
\]

Also, \( S^n \) is defined recursively as follows: \( S^n = S^{n-1} \cup h' \), where

\[
h' := \arg\max_{h' \in S^{n-1}} \sum_{h \in S^{n-1}} (\beta^h)^2 + \left( \beta^{h'} \right)^2 - n \left( \frac{1}{n} \left( \sum_{h \in S^{n-1}} \beta^h + \beta^{h'} \right) \right)^2.
\]

and \( S^2 := O^2 \). Note that, equivalently,

\[
h' := \arg\max_{h' \in S^{n-1}} \sum_{h \in S^{n-1}} \left( \beta^{h'} - \beta^h \right)^2.
\] (30)

Lemma A . 1 There exists an \( r_o^n < n \) such that \( O^n \) consists of the first \( r_o^n \) agents and the last \( n - r_o^n \) agents; i.e., \( O^n = \{1, \ldots, r_o^n, H - (n - r_o^n) + 1, \ldots, H\} \). Moreover, if we let \( \beta_o^n = \frac{1}{n} \sum_{h \in O^n} \beta^h \), then

\[
\beta_o^n < \beta_o^n < \beta^{H-(n-r_o^n)+1}.
\] (31)

Proof of Lemma A . 1 By contradiction. Suppose the statement does not hold. Pick a couple \( s_1, s_2 \in O^n \) such that \( \beta^{s_1} \leq \beta_o^n < \beta^{s_2} \). Then either

Case 1: \( \{\beta^h \mid h \in O^n, h \leq s_1\} \) are not the \( r \) smallest betas, for some \( r \leq s_1 \); or

Case 2: \( \{\beta^h \mid h \in O^n, h > s_2\} \) are not the \( n - r \) largest betas, for some \( r \leq s_1 \) (i.e., \( (n - r) > s_2 \)); or finally

Case 3: both the above cases hold.

We will now show that in either of these cases it is possible to pick an agent \( k \) and substitute him with another agent \( k' \) and improve welfare.

The change in welfare due to the substitution of \( k \) with \( k' \) can be calculated to be:

\[
n \sum_{h \in O^n - \{k\} \cup \{k'\}} (\beta^h)^2 - \left( \sum_{h \in O^n - \{k\} \cup \{k'\}} \beta^h \right)^2 - n \sum_{h \in O^n} (\beta^h)^2 + \left( \sum_{h \in O^n} \beta^h \right)^2
\]
which in turn, after some algebra, can be written as:

\[(\beta^k - \beta^k) \left( (n-1) \left( \beta^k - \beta^n \right) + 2n \right) \beta^n.\]

By construction, \(\beta^{s_1} \leq \beta^n.\)

In Case 1 then, taking \(k = s_1\) and \(k'\) such that \(\beta^{k'} < \beta^{s_1}\), we can improve welfare. Similarly, in Case 2, taking \(k = s_2\) and and \(k'\) such that \(\beta^{k'} > \beta^{s_2}\), we can improve welfare. In Case 3, welfare can be improved as in Case 1 (and also as in Case 2).

Moreover, (31) is implied by our construction. We have picked in fact \(s_1, s_2\) such that \(\beta^{s_1} \leq \beta^n < \beta^{s_2}\), and we just showed that \(s_1 = r^n_o\) and \(s_2 = H - (n - r^n_o) + 1\).

Lemma A.2 There exists an \(r^n_s < n\) such that \(S^n\) consists of the first \(r^n_s\) agents and the last \(n - r^n_s\) agents; i.e., \(S^n = \{1, \ldots, r^n_s, H - (n - r^n_s) + 1, \ldots, H\}\). Moreover, if we let \(\beta^n_s = \frac{1}{n} \sum_{h \in S^n} \beta^h\), then

\[\beta^n_s < \beta^n < \beta^{H - (n - r^n_s) + 1}.\]  

(32)

Proof of Lemma A.2 By induction. The case \(n = 2\) is trivial. Assume the statement holds for \(n - 1\). Let

\[\tilde{h} := \arg\max_{h' \notin S^{n-1}} \sum_{h \in S^{n-1}} \left( \beta^{h'} - \beta^h \right)^2.\]

But note that \(\sum_{h \in S^{n-1}} \left( \beta^{h'} - \beta^h \right)^2 = (n-1) \left( \beta^{h'} - \beta^{n-1} \right)^2\); and hence \(\tilde{h}\) is either \(r^{n-1}_s + 1\) or \(H - (n - r^{n-1}_s)\).

To prove (32), we first write

\[\beta^n_s = \frac{1}{n} \left( (n-1)\beta^{n-1}_s + \beta^{\tilde{h}} \right),\]  

(33)

and consider three cases.

Case 1: \(\beta^{\tilde{h}} < \beta^{n-1}_s\). Then, by the induction hypothesis, \(\beta^{H - (n - r^{n-1}_s) + 1} > \beta^{n-1}_s\), and hence, using (33), \(\beta^{n-1}_s > \beta^{\tilde{h}} > \beta^{\tilde{h}}\), which implies (32).

Case 2: \(\beta^{\tilde{h}} > \beta^{n-1}_s\). Then, by the induction hypothesis, \(\beta^{r^{n-1}_s} < \beta^{n-1}_s\), and hence, using (33), \(\beta^{n-1}_s < \beta^{\tilde{h}} < \beta^{\tilde{h}}\), which implies (32).

Case 3: \(\beta^{\tilde{h}} = \beta^{n-1}_s\). This case can only occur if \(n = H\), which is excluded. \(\diamond\)

Let \(W^n_o := \sum_{h \in O^n} \left( \beta^h \right)^2 - n \left( \overline{\beta}^n_o \right)^2\), and \(W^n_s := \sum_{h \in S^n} \left( \beta^h \right)^2 - n \left( \overline{\beta}^n_s \right)^2\).
Lemma A. 3 \( W_o^n = W_s^n \).

Proof of Lemma A. 3 By induction. \( O^2 = S^2 \) by definition. Assume the statement holds for \( n - 1 \). Let \( O^n_{n-1} \subset O^n \) denote a possible \((n - 1)\)-dimensional subset of \( O^n \).

By the induction hypothesis,

\[
\sum_{h,h' \in S^{n-1}} (\beta^h - \beta^{h'})^2 \geq \sum_{h,h' \in O^n_{n-1}} (\beta^h - \beta^{h'})^2, \quad \forall O^n_{n-1} \subset O^n.
\]  

(34)

Also, by the definition of \( S^n \), either \( S^n = S^{n-1} \cup \{r^n - 1\} \) or \( S^n = S^{n-1} \cup \{H - (n - r^n - 1)\} \).

We consider only the first case; the second is symmetric and is left to the reader: \( r^n = r^n - 1 + 1 \).

Then again by the definition of \( S^n \), and (30),

\[
\sum_{h \in S^{n-1}} (\beta^{r^n} - \beta^h)^2 \geq \sum_{h \in S^{n-1}} (\beta^{h'} - \beta^h)^2, \quad \forall h' \notin S^{n-1}.
\]  

(35)

Let \( O^n := O^n_{n-1} \cup \{\hat{h}\} \). We now write

\[
W^n_s = \sum_{h,h' \in S^{n-1}} (\beta^h - \beta^{h'})^2 + \sum_{h \in S^{n-1}} (\beta^{r^n} - \beta^h)^2, \quad \text{and}
\]  

(36)

\[
W^n_o = \sum_{h,h' \in O^n_{n-1}} (\beta^h - \beta^{h'})^2 + \sum_{h \in O^n_{n-1}} (\beta^{\hat{h}} - \beta^h)^2.
\]  

(37)

But by (34), the first term of (36) is greater than the first term of (37). We now proceed by picking \( h \in O^n \) such that the second term of (36), also, is greater than the second term of (37).

We can first simplify

\[
\sum_{h \in S^{n-1}} (\beta^{r^n} - \beta^h)^2 - \sum_{h \in O^n_{n-1}} (\beta^{\hat{h}} - \beta^h)^2 \text{ to}
\]

\[
(n - 1) \left( (\beta^h - \beta^{r^n - 1})^2 - \left( \beta^{r^n} - \frac{1}{n-1} \sum_{h \in O^n_{n-1}} \beta^h \right)^2 \right).
\]

Then, using (35),

\[
\sum_{h \in S^{n-1}} (\beta^{r^n} - \beta^h)^2 - \sum_{h \in O^n_{n-1}} (\beta^{\hat{h}} - \beta^h)^2
\]
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\[
\geq (n-1) \left( \beta^h - \overline{\beta}_s^{n-1} \right)^2 - \left( \beta^h - \frac{1}{n-1} \sum_{h \in O_{n-1}} \beta^h \right)^2, \quad \text{if } \hat{h} \notin S^{n-1}.
\]

Finally,

\[
(n-1) \left( \beta^h - \overline{\beta}_s^{n-1} \right)^2 - \left( \beta^h - \frac{1}{n-1} \sum_{h \in O_{n-1}} \beta^h \right)^2 = (n-1) \left( \frac{1}{n-1} \sum_{h \in O_{n-1}} \beta^h - \overline{\beta}_s^{n-1} \right) \left( \beta^h - \frac{1}{n-1} \sum_{h \in O_{n-1}} \beta^h \right)^2.
\]

(38)

We are now ready to choose \( \hat{h} \notin S^{n-1} \) so as to show that (38) > 0. There are several cases.

**Case 1:** \( \beta^r_o = \beta^s_r \). In this case, trivially, \( O^n = S^n \).

**Case 2:** \( \beta^r_o < \beta^s_r \leq \overline{\beta}_s^{n-1} \). In this case, we pick \( \hat{h} = r^n_o \) (note that \( r^n_o \notin S^{n-1} \)). Then,

\[
\beta^r_o < \beta^s_r < \frac{1}{n-1} \sum_{h \in O_{n-1}} \beta^h \leq \overline{\beta}_s^{n-1},
\]

which directly implies that (38) > 0.

**Case 3:** \( \beta^r_o < \beta^s_r \). By the definition of \( S^n \), then, \( \beta^r_o < \beta^s_r < \overline{\beta}_s^{n-1} \). In this case, we pick \( \hat{h} = H - (n-r^n_o) + 1 \).

We first show by contradiction that \( H - (n-r^n_o) + 1 \notin S^{n-1} \). Suppose \( H - (n-r^n_o) + 1 \in S^{n-1} \). Then \( r^n_o < \hat{h} = H - (n-r^n_o) + 1 \leq r^n_{s-1} \). But then, (34) and (35), imply that \( \overline{\beta}_o < \beta^h < \beta^s_r < \overline{\beta}_s^{n-1} \); a contradiction, since in such construction \( O^n \) contains some elements in common with \( S^n \) and all other elements greater than the remaining elements in \( S^n \).

The choice of \( \hat{h} = H - (n-r^n_o) + 1 \) implies that

\[
\beta^h > \overline{\beta}_o > \frac{1}{n-1} \sum_{h \in O_{n-1}} \beta^h \geq \overline{\beta}_s^{n-1},
\]

which in turn implies that (38) > 0.

**Case 4:** \( \beta^r_o > \beta^s_r \). This case is not possible. Otherwise, by Lemma 1, \( \overline{\beta}_o > \beta^r_o \). But \( r^n_o \geq r^n_s \), and hence \( \beta^m_n > \beta^s_r \), which is not possible since in such construction \( O^n \) contains some elements in common with \( S^n \) and all other elements smaller than the remaining elements in \( S^n \). ♦

Proposition 8 is proved in the text.
References


