Gold Coins in a Fiat Currency

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Abstract

We investigate the feasibility of minting gold coins with a face value in excess of their bullion value, to circulate as currency alongside paper money. The problem is modelled in a partial equilibrium setting where individuals hold money because they face unpredictable liquidity shocks. Gold coins have a positive expected return. They are less costly to hold than money and are used to meet large, infrequent shocks. An issue of gold coins increases total money demand. Depending on the nature of the liquidity shocks, the seignorage gains to Government from the increase in money holdings may exceed the cost of the gold used in the coins.

JEL classification: E410, E420, E510
The importance of gold in the world monetary system has declined sharply over the last century. Yet Governments now hold more than twice as much gold as they held at the peak of the Gold Standard, and these holdings have not greatly changed over the last half century\(^1\). The need for holding this gold has been widely questioned. This paper examines the possibility of putting some of the gold back to monetary use in the form of gold coins, circulating in parallel with fiat money rather than in a monetary system tied to the gold standard.

Our main finding is that gold coins, issued with a face value in excess of their bullion value, would circulate as money, playing a distinctive role in meeting large liquidity shocks which are too infrequent to justify the holding of cash. The issue of gold coins would lead to a reduction in cash holdings, but overall holdings of money (cash and gold coins together) would rise, thereby leading to increased seignorage for the authorities. We derive the conditions under which the increase in seignorage would exceed the cost of the gold used.

The history of gold coins goes back over two thousand years\(^2\). A century ago it was at the centre of the international monetary system when most currencies were tied to gold. The economic stresses caused by the First World War and its aftermath greatly weakened the Gold Standard. The last formal link between a major currency and gold finally ended in 1971 when the US Government ceased guaranteeing the conversion of the dollar into gold at a fixed rate of $35/ounce.

But gold has not lost all its links with the monetary system. Gold still accounts for 12% of official sector reserve assets. 33,000 tonnes, with a market value of $300 billion, is held by the official sector (including inter-Governmental bodies); this amounts to some 25% of all the gold which has ever been mined. Most of this gold is held in physical

\(^1\) In 1913, monetary gold, meaning Central Bank or Treasury stocks, together with gold coin in circulation amounted to 11,500 tonnes (Green (1999)). By 2000 the total was 28,800 tonnes having reached a peak of 35,000 tonnes in 1950 (World Gold Council (2001)).

\(^2\) The first use of gold coins is traditionally ascribed to the Lydians in the sixth century BC (see Davies, 1996, p. 62).
form, with less than 5,000 tonnes being lent to the market (Gold Field Mineral Services, 2001).

The use and possible sale of this stock of gold has been the subject of much discussion. Henderson et al (1997) argue that the immediate disposal of official sector gold would have major welfare benefits. Gold sales by the official sector have reduced holdings by around 1% per year over the period 1990/99. Analysts often cite the official sector overhang as a major depressant on the price of gold (see for example Veneroso (1998)). The Washington Agreement of 26 September 1999 between fifteen European Central Banks, with the support of the American and Japanese Governments, implicitly recognises that fears of sales of official sector gold holdings could destabilise the gold market (European Central Bank, 1999). It sets limits both on gold sales and on gold lending by the signatories for a period of five years. While it states that “gold will remain an important element of global monetary reserves”, it does not explain the precise role that gold is intended to play. If these substantial reserves of gold are not to be sold or lent, it is worth exploring whether there is any scope for their use as currency.

The idea of issuing gold coins in a fiat currency is not new. In 1998 the European Parliament proposed that a 100 euro gold coin be issued for general circulation. The purpose of this was to strengthen popular support for the new currency (European Parliament (1998), C4-0597/98). The proposal specified that the coin should have a total cost of production, including the cost of the gold, which was below the coin’s face value. This sharply distinguishes the proposed coin from bullion coins which have a gold content far in excess of their face value, which are not intended for use as currency but are sold to collectors and investors, and are traded on the basis of their gold value.

3 The proposal was rejected by the Council on the grounds that that the coin would not trade at face value, and that it would be hoarded (European Parliament, A4-0044/99).

4 The American Eagle for example, has a legal tender value of $50/oz of gold, less than one fifth of its bullion value. Evidence that such coins are still seen as having some monetary use comes from the dramatic growth in demand for gold coins in the US in advance of the millennium. With fears of disruption in the financial system due to the “Millennium Bug”, gold used in US coin fabrication went up from 16.8 tonnes
This paper is concerned with what might be called currency coins: coins which, at least at the point of issue, have a gold content worth substantially less than their face value. Such coins might reasonably be expected to trade at close to their face value, and thus to have a role as currency. The circulation of such coins in parallel with conventional money raises a number of interesting questions. Would Gresham’s Law, that bad money drives out good, apply, so that the good money (i.e. coins containing gold) is hoarded, and the bad money (i.e. paper money) is used for transactions? How would the total quantity of money in circulation be affected? Would seignorage, the gain to the authorities from issuing liabilities which pay no interest, be fully offset by the cost of the gold used to mint the coins? What would the welfare effects be?

While there is little discussion in the academic literature about the use of commodity money in a fiat currency, the parallel circulation of different forms of money has long excited the interest of economists. Specifically, the controversy over bimetallism – the simultaneous use of gold and silver coin – which was particularly intense towards the end of the nineteenth century, spawned a substantial literature (see for example Darwin (1898), Fisher (1911), Giffen (1892), Laughlin (1896) and Shaw (1894)). The debate over bimetallism has been revived recently by Dowd (1996), Flandreau (1996) and Velde and Weber (2000). Much of the debate has revolved around the sustainability of bimetallism, and whether bimetallism necessarily degenerates into alternating monometallic money standards as the relative market price of the two metals varies around the mint price level.

The debate over bimetallism can be seen as part of a wider debate on whether it is ever possible to have two types of money in circulation simultaneously, or whether, as Gresham’s Law suggests, the bad or less valuable money always drives out the good. “Good” and “bad” money in this context may be gold and silver, newly minted and worn coins, good coins and clipped coins, issues by different mints or monetary authorities, or coins and bank notes. Rolnick and Weber (1986), noting a number of historical exceptions to Gresham’s Law in the US in the nineteenth century and in the UK in the

per year in the quinquennium 1993-97 to 57.2 tonnes in 1998, 60.3 tonnes in 1999 and then down to 5.4 tonnes in 2000 (GFMS (2001, p 98)).
seventeenth century\textsuperscript{5}, argue that good and bad money will coexist, with bad money trading at face value and good money trading at a premium. Selgin (1996) emphasises the critical importance of the legal tender laws, their severity and enforcement, in determining whether two forms of money coexist or not.

In the present paper, the legal tender rules are tight and appear therefore to be tilted in favour of Gresham’s Law. When cash is required for a transaction, the transaction always takes place at face value, whether the cash is in the form of paper or gold. The two parties are not allowed to negotiate a premium for gold coin. At first sight, this appears to ensure that gold coins are never used as money, because people will prefer to pay debts in paper money, which has no intrinsic value, rather than gold which is or may become valuable.

Despite this, there is a role for gold coins in our economy. Holding paper money has an opportunity cost. It pays no interest and does not appreciate. Since the expected return on gold coins is positive, the opportunity cost of holding money in the form of gold coin is less than of holding paper money. It is worth holding gold coins to meet large liquidity shocks which are too infrequent to justify holding paper money. Gold coins circulate less rapidly than paper money, but they are still used as money.

We have departed somewhat from the standard way of modelling commodity money. Much of the formal modelling of economies with two forms of money (for example Velde and Weber (2000), Sargent and Velde (1997), or Sargent and Smith (1997)) is based on Lucas’s (1982) cash-in-advance model. The problem is modelled in a general equilibrium, discrete time setting where identical infinitely long-lived utility-maximising individuals face endowment shocks. Money is needed because consumption goods can only be paid for using money; individuals cannot liquidate assets within a period to meet consumption demands.

The general equilibrium setting is excessively elaborate for our purposes. We are interested in the effect of introducing a finite (and generally small) supply of commodity

\textsuperscript{5} It should be mentioned, however, that Greenfield and Rockoff (1995) dispute the historical evidence.
money into what is predominantly a fiat currency system. Commodity money does not
induce a radical change in the individual’s opportunity set. With or without the
commodity money, individuals can save by holding bonds, they can acquire liquidity by
holding cash, and they can get exposure to the commodity price by buying the
commodity or trading on a commodity forward market. We use a partial equilibrium
model where the general price level, the level of interest rates, and the per capita level of
Government debt are all determined exogenously.

The model is set in continuous time. This simplifies the exposition and also has the
advantage that no arbitrary restriction is imposed on the maximum velocity at which
money circulates. Goods and financial claims can be bought or sold at any time, but they
need to be paid for in money. The stock of money needed to meet known future
consumption requirements is infinitesimal since assets can be sold immediately before
the need arises. Stocks of money are needed only to meet shocks which are unforeseen.

We represent the cash-in-advance constraint in reduced form rather than by a formal
model of the constrained consumption decision. In a Lucas economy, an individual with
large cash holdings, and who is therefore unrestricted by the cash-in-advance constraint,
optimally chooses to spend an amount $q$ in cash when faced with an unexpected shock.
The individual who has no cash is constrained; the constraint ensures that there is a
shadow price $c$ on cash, where $c > 1$. In our model, the individual faces exogenous shocks
which cost $q$ to meet if he has at least $q$ in cash, and $qc$ to meet if he has no cash. This
representation is consistent with a variety of interpretations of the nature of these shocks;
the shocks could be endowment shocks (as in most of the literature) or shocks to the
utility function, or simply unforeseen demands for immediate cash payment$^6$.

$^6$ Feenstra (1986) demonstrates the equivalence of a number of alternative methods of modelling money.
One natural alternative approach to modelling liquidity demand is to assume that money which is legal
tender generates a flow of liquidity services. We show in Appendix B that given certain assumptions about
the dependence of this flow on the explanatory variables, the cash-in-advance used here and the flow of
liquidity services approaches are equivalent.
In another departure from the literature, we assume that while individuals are identical, the shocks they face are idiosyncratic. There are no shocks to consumption or endowment at the aggregate level. The only source of uncertainty at the aggregate level concerns the price of the commodity (gold). With the assumed existence of deep and liquid forward markets in the commodity, households can hedge commodity price risk. The only risks in the economy are therefore either idiosyncratic or hedgeable. The assumption that households are risk neutral, which we make for tractability, is therefore innocuous.

Our basic model is set out in Section I, where the only form of money is paper money, and the only other financial asset is Government bonds. Gold and gold coins are introduced in section II. The third section examines the magnitude of the gains to Government and to households from issuing gold coins. In Section IV the model is parameterised and solved numerically. The final section concludes.

I. AN ECONOMY WITHOUT GOLD COINS

In this Section we describe our model of the supply and demand for money. We take the size of household financial assets as given. The only decision for households is to determine how much of their assets to hold as cash and how much as interest-paying bonds. The Government supplies as much money as households choose to hold. We show how the demand for money depends on the size and frequency of liquidity shocks and the rate of interest.

A. The Demand for Money

The economy consists of two sectors: Government and Households. There is a continuum of identical households. Households are risk neutral. They have savings which they can hold either in the forms of Government bonds which pay interest at some fixed rate $r$, or in the form of money which pays no interest. They hold money because they face unpredictable liquidity shocks. A liquidity shock can be thought of as a liability falling due unexpectedly, or an unforeseen demand for some good or service which is required immediately and cannot be bought on credit. The magnitude of the shock (denoted by $q$)

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is the cash sum required. If the household is holding money with a face value of \( q \) or more, it can meet the shock in full by paying \( q \).

If the household does not have any cash when the shock occurs, it suffers a utility loss with a cash value of \( qc \) where \( c > 1 \). Two possible interpretations of \( c \) (the severity of the shock) come to mind: it could represent \( 1/(1-a) \) where \( a \) is a proportional transaction cost associated with an unplanned sale of bonds to meet the required payment; alternatively \( qc \) can be thought of as the utility which the household would have received from the good or service, but which it has to forego because it does not have the cash to pay for it.

A liquidity shock is thus characterised by the pair \((q, c)\). If the household has sufficient cash, the cost of the shock is just \( q \). If it has no money, the cost is \( cq \). A shock can be met in part by cash. If the household has money \( x (\leq q) \), the cost of the shock is:

\[
x + (q - x)c.
\]  

(1)

The household has to decide its optimal mix of money and bonds; the more money it holds, the cheaper it is to respond to liquidity shocks, but the lower the return the household earns on its financial assets.

B. The Supply of Money

The Government has a financing requirement \( F \) per household. To finance itself it issues \textit{money} which pays no interest and can be used to pay for goods and services, and \textit{bonds} which pay interest at a fixed rate \( r \). The Government has a monopoly on the issue of money. The Government does not seek to control the supply of money. It stands ready to exchange money for bonds and bonds for money at par.

Any reduction in interest income to households collectively caused by an increase in money holdings is recycled by the Government back to households through some non-distorting tax rebate, so the financing requirement \( F \) and the rate of interest \( r \) are not affected. \( F \) and \( r \) are assumed to be constant.
C. Liquidity Shocks

The demand for money is driven by the existence of liquidity shocks which are identically and independently distributed across households and over time. The intensity of shocks of magnitude $q$ and severity $c$ is $\lambda(q, c)$.

We make the following assumptions about $\lambda$:

[A1] $\lambda$ is continuous and strictly positive throughout the domain $(0, \infty) \times (1, \infty)$;

[A2] $\int_{c=1}^{\infty} \int_{q=0}^{\infty} \{c-1\} \lambda(q, c) dq \ dc > r$.

[A3] $\int_{c=1}^{\infty} \int_{q=q^*}^{\infty} \{c-1\} \lambda(q, c) dq \ dc < r$.

These assumptions are technical. Assumption [A1] ensures the demand function for money is strictly monotonic. Assumption [A2] ensures that the demand for money at the current interest rate is positive and finite, while assumption [A3] ensures that it does not exceed the Government’s financing requirement $F$.

D. Solution to the Model

With the Government supplying whatever money is needed, the money supply is determined entirely by household demand. Denote the per capita money stock by $q^*$. A household will hold $F - q^*$ in bonds and receive interest on this. Whenever it faces a liquidity shock $(q, c)$ it will meet it in full in cash if it can. Thus the flow of interest income less liquidity costs will be:

$$\left(F - q^*\right) r - \int_{c=1}^{\infty} \left[\int_{q=q^*}^{\infty} q \lambda(q, c) dq + \int_{q=q^*}^{\infty} q^* \left(\lambda(q, c) dq - (q - q^*) c \lambda(q, c) dq\right) dc\right].$$

This is maximised by choosing $q^*$ to satisfy the first order condition:

$$\int_{c=1}^{\infty} \int_{q=q^*}^{\infty} \{c-1\} \lambda(q, c) dq \ dc = r.$$
The existence of a unique viable internal solution ($0 < q^* < F$) is guaranteed by Assumptions [A1] – [A3]. The second-order conditions are necessarily satisfied since $\lambda$ is positive.

Equation (3) is a demand function for money, relating per capita money demand $q^*$ to the interest rate $r$. The demand function is downward sloping, as one would expect. With higher interest rates, households hold less cash and more bonds.

We have no need to model inflation, but it is useful in interpreting the model to understand which quantities are real and which are nominal. It is natural to interpret $\lambda$ as representing the distribution of real shocks; the interest rate $r$ is the cost of holding cash and so is the nominal interest rate. Thus equation (3) expresses the demand for real cash balances as a function of the nominal interest rate.

II. THE ECONOMY WITH GOLD COINS

In this section, we introduce gold coins. We show that gold coins coexist with paper money, that they are used only to meet the largest and most severe liquidity shocks, and that in general they trade at a premium to their face value.

A. Gold Coins

Assume that, in addition to paper money (“cash”), the Government mints gold coins. Both types of money are legal tender and can be used at face value (which we normalize to 1) for paying for goods and services and bonds. Gold coins contain a fixed quantity of gold. The cost of minting coins (turning gold bullion into gold coins) and of melting down gold coins (turning coins into bullion) is zero. Anyone may turn gold coins into bullion; only the Government can turn gold bullion into coins.

Assume that the Government mints a fixed number of gold coins and credibly commits to mint no further gold coins. So long as it holds gold coins, the Government stands ready to sell them for cash at their face value. The Government commits to redeem gold coins for cash at any time at face value.
Gold coins are intrinsically more valuable than paper money since, if the price of gold rises sufficiently, they can be melted down and sold for their gold content at a premium to their face value. Households may trade gold coins with each other; these transactions do not have to occur at face value. Indeed, as will be demonstrated, the price $P$ at which the coins are traded is generally but not invariably greater than 1.

Households allocate their financial resources between three types of financial asset: cash which pays no interest but is well suited for meeting liquidity needs, bonds which pay interest but cannot be used to meet liquidity needs, and gold coins. Gold coins occupy an intermediate position between cash and bonds; they can be used as money, but it is costly to do so since legal tender rules require that they be exchanged at face value rather than market value.

B. The Gold Market

Gold bullion is traded in a deep and liquid market. By normalising suitably, we can use the variable $G$ to represent both the price of gold and the market value of the gold content of a coin with a face value of 1. $G$ is exogenous to the model. It is assumed to follow a Markov process; the precise form of the process is not important at this stage.

Gold that is not used for money can be used for jewellery. It is costless to turn gold into jewellery, and to melt jewellery and turn it into gold. Jewellery gives utility to its owner, so there is a market for borrowing or lending gold at a rate $l$, the gold lease rate, which is assumed to be constant.

With a spot market, and frictionless borrowing and lending markets for cash and gold, households can hedge gold price risk perfectly. Households that hold gold coins can eliminate gold price risk since they can sell gold forward. Conversely, households that want exposure to the gold price can get it through the forward market whether the Government issues gold coins or not. With households being risk neutral, the expected
return on the gold forward price must be zero; otherwise households would wish to hold unbounded gold forward positions.\(^7\)

The model is stationary; the Markov assumption ensures that the only state variables are the gold price \(G\) and the per capita supply of gold coins, \(k^g\). The Government sets \(k^g\) initially. It can never increase, but it can decline if coins are melted down. The state of the world can therefore be represented by the pair \((G, k^g)\) and the price \(P\) of a gold coin can be written as a function \(P(G, k^g)\). Gold coins can be surrendered for cash or bonds at their face value, and they can be melted down, so \(P \geq 1\), and \(P \geq G\).

C. Solution to the Model

The total expected return on holding a gold coin is the sum of the expected financial return and the return in the form of liquidity services. For there to be equilibrium in the market for gold coins, either the total expected return is equal to the riskless interest rate (so households are happy to hold the gold coins), or the total expected return is less than the riskless rate and the price of gold coins is \(G\) (so the coins are melted down).

To avoid bubble solutions, we impose the additional requirement that if the price of gold is high enough, the face value of the gold coin becomes irrelevant, and its price tends to its gold value. More formally, we assume:

\[\text{[A4]} \quad P(G, k^g) - G \text{ tends to zero as } G \text{ tends to infinity.}\]

Let \(G^M(k^g)\) denotes the lowest price at which it is optimal to melt gold coins. We prove in Proposition 1 below that \(G^M\) is finite. When melting is optimal, gold coins are worth their gold value, so \(P(G, k^g) = G\) for all \(G \geq G^M(k^g)\).

Suppose \(P > 1\), and a household is holding \(x\) in cash and \(y\) in gold coins. Faced with a liquidity shock \((q, c)\), the optimal strategy is first to use cash to meet the shock and, if

\(^7\) Alternatively, if households were risk averse, the valuation equations below would use the risk adjusted probability measure rather than the objective measure, and the results would be the same.
there is insufficient cash, and the shock is severe enough, use gold coins. More formally, from equation (1), the cost $f$ of the liquidity shock if it occurs when the household is holding $x$ in cash and $y$ in coins is:

$$f(q, c; x, y) = \begin{cases} 
q & \text{if } q \leq x \\
q + (q - x)c & \text{if } q > x \text{ and } c \leq P \\
x + (q - x)c & \text{if } q \in (x, x + y) \text{ and } c > P \\
x + yP + (q - x - y)c & \text{if } q \geq x + y \text{ and } c > P.
\end{cases} \quad (4)$$

By holding $q^c$ in cash rather than bonds, the return on financial assets is reduced by $q^c r^c$ per unit time. By holding $q^g$ in gold coins, the expected return is reduced by $q^g r^g$ per unit time where:

$$r^g = rP - E[dP/dt]. \quad (5)$$

$r^g$ is the opportunity cost per unit time of holding one gold coin. Households choose $q^c$ and $q^g$ so as to minimise the sum of the cost of meeting liquidity shocks and the return foregone by holding money rather than bonds. Formally, they solve the following problem:

$$\text{Min } q^c, q^g \left[ . \int f(q, c; q^c, q^g) \lambda(q, c) dq dc + q^c r + q^g r^g \right] \quad (6)$$

subject to $q^c \geq 0$ and $q^g \geq 0$.

The first order conditions for problem (6) are:

$$\int_{c=1}^{\infty} \int_{q=q^c}^{\infty} \{ \text{Min}(c, P) - 1 \} \lambda(q, c) dq dc = r - r^g. \quad (7a)$$

$$\int_{c=P}^{\infty} \int_{q=q^c + q^g}^{\infty} \{ c - P \} \lambda(q, c) dq dc = r^g. \quad (7b)$$

8 There is no reason, given the Poisson arrival of shocks, to retain some cash against a second shock. There is time to liquidate bonds and rebuild cash balances before a second shock arrives.
The left hand side of (7a) is the reduction in liquidity costs from substituting a gold coin with cash; the right hand side is the corresponding reduction in financial return. (7b) equates the marginal benefits and costs of substituting a gold coin for a bond. Equations (7a) and (7b) determine the demand for cash and gold given the price of gold coins ($P$) and the cost of holding them ($r^g$). Any coins in existence which are not held by households to meet liquidity needs are held by Government and can be bought at face value. So either $q^g < k^g$ and $P = 1$, or $q^g = k^g$.

The nature of the equilibrium depends on the number of gold coins in existence, $k^g$. In particular, it depends on whether the face value of the coins is greater or less than the demand for cash in an economy without gold coins ($q^*$, determined by (3) above). We look in turn at an economy with a small number of gold coins ($k^g < q^*$), and at an economy with an abundance of gold coins ($k^g > q^*$).

D. Results in An Economy with a Small Issue of Gold Coins ($k^g < q^*$)

**Proposition 1.** If the number of gold coins issued is small (i.e. if $k^g < q^*$, the issue is less than the amount of money which would be held in an economy without gold coins), and so long as the gold price remains below $G^M$, there is an equilibrium with the following features:

(i) Cash and gold coins coexist and both have a liquidity function.

(ii) All gold coins are held by households ($q^g = k^g$).

(iii) The market price of gold coins $P$ is strictly greater than their face value ($P > 1$).

(iv) Households use cash first, reserving gold coins to meet shocks which are both large ($q > q^c$) and severe ($c > P$).

(v) The opportunity cost of holding a gold coin ($r^g$) is strictly positive, but strictly less than the cost of holding the same face value in cash.

(vi) Less cash is held than in an economy without gold coins, but the total amount of money held (cash plus gold coins) is greater ($q^c < q^* < q^c + q^g$).

(vii) If $G$ rises above $G^M$ some or all of the coins are melted down.
(viii) The amount of cash held by households, $q^c$, is defined implicitly by:

$$\int_{c=1}^{q^c} \int_{q=1}^{q^c} \{c-1\} \lambda(q,c) dq \ dc = \int_{c=1}^{q^c} \int_{q=1}^{q^c+k^g} \{c-P\} \lambda(q,c) dq \ dc.$$  

(8)

**Proof.** See Appendix.

The proof demonstrates that the first order conditions (7a and 7b) can always be satisfied with strictly positive values for demand for cash and gold coins, and that the second order conditions are also satisfied. Equation (8) says that the flow of liquidity services from incremental cash must be identical in a world with or without gold coins; the two flows must equal the riskless interest rate.

Gold coins have a market value in excess of their face value. Households do not use them to meet their liquidity needs unless they have no cash left. In that sense, gold coins are hoarded and are not in free circulation. If a household receives a gold coin in payment for services, it will either keep it, or sell it at market value, which is at a premium to face value. Any gold coins received by the Government are immediately acquired by households in exchange for cash at face value.

However, gold coins ("good money") still retain the attributes of money. They are used to meet liquidity shocks when the shocks are both large and severe. Because of this liquidity service which gold coins provide, they do offer an expected financial return which is less than the interest rate. Because they are more costly to use for meeting liquidity needs than cash, their expected return, unlike that of cash, is strictly positive.

It is useful to explore the limit case as the issue of gold coins gets infinitely small:

**Corollary.** As $k^g$ tends to zero, $q^c$ tends to $q^*$, and $G^M(k^g)$ tends to $G^M < \infty$. So long as $G < G^M$ the reduction in cash holdings for each gold coin issued is given by:

$$\lim_{k^g \rightarrow 0} \left[ \frac{q^* - q^c}{k^g} \right] = \frac{\int_{c=1}^{q^*} \{c-P\} \lambda(q^*,c) dc}{\int_{c=1}^{q^c} \{c-1\} \lambda(q^*,c) dc}.$$  

(9)

Once $G$ has reached $G^M$ the gold coin is melted down and thereafter $q^* = q^c$. 

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**Proof.** The first sentence follows immediately from Proposition 1. Equation (9) is the limiting case of equation (8).

If the market price of gold coins is close to their face value (i.e. as $P \to 1$), the right hand side of (9) approaches 1, gold coins and cash are close substitutes, and the issue of gold coins is fully offset by a reduction in cash holdings. If the market price of the coins is so high that they are rarely used for meeting liquidity shocks (i.e. as $Pr\{c > P\} \to 0$) the right hand side approaches zero, and the issue of gold coins has little impact on the size of cash holdings.

E. Results in an Economy with a Large Issue of Gold Coins ($k^g > q^*$)

We now consider what happens if the Government mints so many gold coins that the supply exceeds the demand for paper money in the absence of gold coins.

**Proposition 2.** If $k^g > q^*$, and so long as the gold price remains below the level at which coins are melted down ($G < G^M$), there are three distinct regions. They are:

1. **Low gold price:** both gold coins and cash circulate freely. Gold coins trade at their face value ($P=1$). The total amount of money in circulation (gold coins and cash) is $q^*$. The Government has gold coins which it is ready to exchange for cash.

2. **Medium gold price:** no cash is used. Gold coins are the only money. The amount of money in circulation is $k^g$. Gold is traded at a premium to face value ($P>1$).

3. **High gold price:** both cash and gold are used as money, but gold coins are traded at a premium, and the equilibrium is similar to that with a small issue of gold coins.

**Proof.** See Appendix.

In the low gold price region, there are so many gold coins, and the face value of the gold coins so far exceeds their bullion value, that households do not want all the gold coins in issue. Some gold coins are left with the Government. The Government is ready to exchange cash for coins at par, so the coins trade at par and households are indifferent between cash and coins.
With the medium gold price, the potential for appreciation means that gold coins have a strictly positive return and they dominate cash. Anyone facing a liquidity shock who happens to be holding cash will pay in cash rather than gold coins. But the expected return on holding gold coins is so much higher than the expected return on holding cash that households do not include any cash in their portfolio. One could view this as a counter-example to Gresham’s Law – there is so much good money (gold coins) available that it is driving out bad.

While the price of gold coins is continuous at the point of transition between the first two regions, the expected return on holding coins and the demand for gold coins is not. Below the transition point the expected return on coins is zero, while above it the expected return is strictly positive. This return is sufficient to increase the demand for gold coins from $q^*$ to $k^2$. All the gold coins are bought by households; none remain with Government. Their price therefore can rise above par as the gold price rises.

In the high gold price region, the price of gold coins is so high that they are costly to use for liquidity purposes, and cash starts to have a distinctive role, particularly in meeting the less severe liquidity shocks. The equilibrium is similar to that which obtains with a small issue of coins.

### III. GAINS FROM MINTING GOLD COINS

In this section we consider the gains from issuing gold coins. We focus on the case where the issue of coins is small. We derive expressions for the magnitude of gains computed from the Government’s perspective, and also from the perspective of households. Since any gains to Government are assumed to be redistributed to households in the form of a non-distorting tax rebate, the gains to households include the gains to Government.

We show that, provided the size and severity of liquidity shocks are independent, the costs to Government of issuing gold coins exceed the benefits. Households get direct benefits from the gold coins in terms of increased liquidity. Whether these are sufficient to ensure the overall gain to households is positive depends on the balance between the
flow of liquidity services provided by the gold coins on the one hand, and the lease rate on gold which represents the costs of tying up the metal on the other.

A. Gains to Government

Two contradictory arguments about whether the Government gains or loses by issuing gold coins suggest themselves: if the Government can manufacture a gold coin at the cost of $G$ and sell it at its face value of 1, then it makes a profit of $1 - G$, which is positive provided only that $G < 1$. On the other hand a Government which wants to issue an additional unit of money could create paper money costlessly. On this reading, gold coins are just high cost money, and the Government loses $G$ on every gold coin it mints.

The arguments differ in their assumptions about the impact of issuing gold coins on the total amount of money in circulation. If the amount of cash in circulation is unaffected by the issue of gold coins, the Government will indeed earn $1 - G$ on each coin issued. If the issue of gold coins causes an exactly offsetting reduction in cash, the Government will lose $G$. We can estimate the reduction in cash resulting from the issue of gold coins, and hence compute the gain or loss to Government.

Since the argument concerns the behavior of the model over time, it is necessary to subscript variables such as the gold price $G$, the price of gold coins $P$, the quantity of cash and coins in circulation ($q^c$ and $q^g$) and the quantity of coins in issue $k^g$, with a subscript $t$ to denote time. $t = 0$ is the time when Government issues the coins, and $t = T$ is the (random) time when the price of gold first reaches $G^M$ and the coins are melted down.

The number of gold coins initially issued is $k^g_0$. By Proposition 1, if $k^g_0 < q^*$ then all the gold coins go straight into circulation. The Government pays $k^g_0 G_0$ for the gold; total money holdings at time $t$ increase by $k^g_0 + q^c_t - q^g_t$, and the Government’s cost of debt servicing declines accordingly. The present value of the gains to Government per coin minted (valued at the point of issue, $t = 0$) can therefore be written as:
\[ \Pi^G = \mathbb{E} \left[ \int_{t=0}^{\infty} \left( 1 - \frac{q_t^s - q_t^c}{k_t^G} \right) e^{-rt} dt \right] - G_0. \]  

(10)

We focus on the limit case as the size of the issue tends to zero.

**Proposition 3.** If the size of liquidity shocks faced by households is independent of the severity of the shock, then the marginal gain to Government is strictly negative. It is:

\[ \Pi^G = -\{P_0 - 1\} - \mathbb{E} \left[ \int_0^T \dot{G} e^{-rt} dt \right]. \]  

(11)

**Proof.** See Appendix.

Equation (11) has a natural interpretation. When the Government mints gold coins at time \( t = 0 \), it issues them at their face value, 1. But the gold coins are immediately traded at a higher price \( P_0 \). The difference between the face value and the market value is one component of the cost of the issue to the Government. The second component in equation (11) is the leasing income foregone because gold that is incorporated in coins cannot be leased and used as jewellery.

If size and severity are independent, proposition 3 shows that seignorage gains are insufficient to pay for the gold content of the coins. In Section IV we show that where the covariance between the size and severity of shocks is sufficiently high the Government can mint coins profitably. The intuition is that in this case gold and paper money become complements rather than substitutes, with cash being reserved for small, frequent shocks, and gold coins being used for rare and severe shocks. Total money holdings are therefore higher and Government seignorage profits increased.

**B. Gains to Households**

We now consider the gains to households from a small issue of gold coins. Coins reduce liquidity costs. But at the margin, the benefit of increased liquidity is equal to the opportunity cost of holding the coins. Since the issue is small there is no consumer surplus, and the net benefit to households of increased liquidity is zero. There are three
other sources of gain to households from the issue of gold coins. First, the gain (positive or negative) to Government $\Pi^G$ which is then redistributed to households. Second, the windfall gain to households at the point of issue; since the coins are worth $P_0$ but are issued by the Government at par, the gain per coin issued is $P_0 - 1$. Third, households gain as creditors when they are paid in gold coin; in return for a debt of 1 they receive a gold coin with a face value of 1 but a market price of $P_t$.

The total gain to households $\Pi^H$ per coin issued (including the indirect gain from Government) is:

$$\Pi^H = \Pi^G + \{P_0 - 1\} + E \left[ \int_0^T \int_{c=P}^{\infty} \int_{q=q^*}^{\infty} (P_t - 1)\lambda(q,c) dq \, dc \, e^{-\eta t} \, dt \right]. \tag{12}$$

Proposition 4. If the size of liquidity shocks faced by households is independent of the severity of the shock, then the marginal gain to Households is given by:

$$\Pi^H = E \left[ \int_0^T \left\{ \left( \int_{c=P}^{\infty} \int_{q=q^*}^{\infty} (P_t - 1)\lambda(q,c) dq \, dc \right) - lG \right\} e^{-\eta t} \, dt \right]. \tag{13}$$

Proof. Follows immediately from (12) and Proposition 3.

Proposition 4 shows that the gain from the issue of gold coins can be represented as the difference between two flows: the first is the value of gold coins as money, as represented by the rate at which they are used multiplied by the difference between their market and face value. The second is the cost of tying up a useful commodity in coins, as represented by the convenience yield. It can readily be seen that the sign depends on the size of liquidity shocks on the one hand (the more severe they are, the greater the welfare gain) and the convenience yield on the other (the higher the yield, the lower the gain). This is illustrated further in the parametric example in the following section.

IV. PARAMETRIC EXAMPLE

We have derived expressions for the price of gold coins, and for the level of gains to Governments and households from issuing coins. In this section we specialise the model
by assuming a particular form for the gold price process and for the nature of liquidity shocks. Using plausible parameter values, we quantify the gains and do a sensitivity analysis.

A. Specification of the Parametric Model

The gold price follows some Markov process. To parameterise the model as economically as possible, we assume:

[A5] \( G \) follows a geometric diffusion with constant volatility \( \sigma \).

Given that households are risk neutral, and that there is equilibrium in the gold lending market, the expected return on gold must equal the difference between the interest rate \( r \) and the lease rate \( l \). So:

\[
dG/G = (r - l) \, dt + \sigma \, dz,
\]

where \( z \) is a standard Brownian process.

Liquidity shocks occur with intensity \( \lambda \). We assume\(^9\):

[A6] The size of shocks \( q \) is distributed negative exponentially.

[A7] The severity \( c \) of the shock is a positive linear function of its size \( q \).

It turns out that we need now concern ourselves only with two aspects of the distribution of liquidity shocks – the expected severity of shocks which can just be met in full, and the expected severity of shocks which cannot be met in full. Denote these by \( P^* \) and \( P^+ \) respectively, so:

\[
P^* \equiv E[c \mid q = q^*] \quad \text{and} \quad P^+ \equiv E[c \mid q > q^*].
\]

\(^9\) While Assumption [A1] is not strictly satisfied, the demand for cash and coins is strictly monotonic, which is all we need.
If the size and severity of shocks are independent, then $P^* = P^*$. 

We can now solve the model in the limiting case where the size of the issue of gold coins tends to zero ($k^g \to 0$). If the gold price is high ($G > G^M$) the coins will be melted down immediately. Equations (5), (10) and (12) express $P$, $\Pi^G$ and $\Pi^H$ as expectations. Having now specified the dynamics of the gold price, we can use the Martingale Representation Theorem to turn the expectations into differential operators. This is done in Proposition 5 below.

Different equations hold depending on whether the gold price $G$ is greater or less than some critical level $G^*$. Below $G^*$, gold coins are always used when shocks are too large to be met fully from cash holdings. Above $G^*$, some shocks which are not severe enough to make it worth using gold coin.

**Proposition 5.** As the size of the issue tends to zero, the price of a gold coin ($P$), and the gains to Government ($\Pi^G$) and to households ($\Pi^H$) per coin, which are all functions of the gold price $G$, satisfy the following differential equations over the range $[0, G^*]$: 

$$
\Delta P + r \frac{P^* - P}{P^* - 1} = 0 \\
\Delta \Pi + r \frac{P - 1}{P^* - 1} = lG \\
\Delta \omega + r \frac{P - 1}{P^* - 1} = lG
$$

(16a)

where $G^* \equiv P^1(P^*)$, and $\Delta$ is the differential operator:

$$
\Delta \equiv \frac{1}{2} \sigma^2 G^2 \frac{d^2}{dG^2} + (r - l)G \frac{d}{dG} - r.
$$

In the range $[G^*, G^M]$, they satisfy the differential equations:
\[
\Delta P + r \frac{K \exp \left[ \left( P^* - P \right) \left( P^* - P^+ \right) \right]}{P^* - 1} = 0
\]

\[
\Delta \Pi^G + r = l G \quad \quad \quad (16b)
\]

\[
\Delta \Pi^H + r \left( 1 + \frac{P - P^*}{P^* - 1} \right) \exp \left[ \left( P^* - P \right) \left( P^* - P^+ \right) \right] = l G
\]

subject to the boundary conditions \( P(0)=1, \Pi^G(0)=0, \Pi^H(0)=0, P(G^M)=G^M, P'(G^M)=1, \)

\( \Pi^G(G^M)=1-G^M \) and \( \Pi^H(G^M)=0. \)

**Proof.** See Appendix.

The parameters of the model now number five: \( r \) the riskless interest rate, \( l \) the gold lease rate, \( \sigma \) the volatility of the gold price, and two liquidity parameters \( P^* \) and \( P^+ \). \( P^* \) is the severity of the shock which can just be met with existing cash holdings. \( P^+ \) is the expected severity of those liquidity shocks which cannot be met in full.

In general, the equation set of Proposition 5 does not admit of analytic solutions, but it is straightforward to solve numerically.

**B. Numerical Solutions**

We solve the model numerically using a range of plausible parameter values. For the base case we take an interest rate \( r \) of 5%/year, a gold lease rate \( l \) of 1%/year, and an annualised volatility of the gold price \( \sigma \) of 15%.

Choosing appropriate values for the liquidity demand parameters \( P^* \) and \( P^+ \) is more difficult. In a reasonably sophisticated economy with automatic teller machines, and debit and credit cards being widely accepted, it seems implausible that the value of holding cash when a liquidity shock occurs exceeds a few percent. We therefore look at three cases. One case \( (P^* = P^+ = 1) \) is an extreme representation of a world where there is little need for money; liquidity shocks are never severe. In the second case \( (P^* = P^+ = 1.05) \) there is still no correlation between the magnitude and the severity of shocks, but there is a benefit to holding sufficient cash when a liquidity shock occurs equal to 5% of the size
of the shock. The third case \((P^* = 1.05 \text{ and } P^+ = 1.10)\) corresponds to a case where large shocks are more severe, and where the shocks which are larger than the household’s cash holdings are on average 5% more severe than those which exactly equal those holdings.

Figure I shows the market value of a gold coin as a function of the gold price (or the gold content of the coin). For all parameter values, the market value is close to face value when the gold price is low, and tends to the gold price when the gold price is high. The price of a gold coin increases with the liquidity parameters \(P^*\) and \(P^+\).

In the extreme case where \(P^* = P^+ = 1\), gold coins are used for meeting liquidity shocks only when their market price is equal to face value. The market price of a coin is identically equal to 1 if the gold price is less than some critical value (0.85 – so the gold content of the coin is worth no more than 85% of face value); the coins are used interchangeably with cash up to this point. If the gold price is higher than this the coins have a market value in excess of their face value and the coins are not used for providing liquidity. If the gold price exceeds about 1.30 \((G^M)\), the optimal strategy is to melt down the gold coin and sell the gold, and the coin trades on its gold value.

The price of the coin would be the same if it were not legal tender, but with the holder retaining the right to sell the coin back to Government at face value any time. The difference in the value of the gold coin between this extreme case and one with more realistic parameters can thus be taken as a measure of the liquidity value of the coin.

Figure II shows the present value of Government gains from issuing the coins expressed as a percentage of the cost of the gold used to mint them \((\Pi^G/G)\) as a function of the gold content of the coins. The graphs are all unimodal. When \(P^* = P^+\), the graphs lie below the axis, as is expected from Proposition 3. The gain to the Government actually declines with \(P^*\): the market value of the coins increases so the Government loses more money when it issues the coins at par, and the coins are less likely to be melted down, ensuring that the loss of lease income is higher. The minimum cost to Government is obtained by issuing coins with a gold value equal to about 88% of face value. With no liquidity shocks \((P^* = 1)\) the minimum cost is 7.3% of the value of the gold used. In the presence
of significant liquidity shocks \((P^* = 1.05)\) the minimum cost increases to 9.9% of the gold value.

The costs to Government reduce significantly if large shocks are also severe shocks. With \(P^* = 1.05\) and \(P^+ = 1.10\), the Government can restrict the cost of issuing gold coins to 1.4% of the value of the gold used; this optimum occurs with a substantially lower gold content (74%) than in the case where the size and severity of shocks are independent.

The cost to Government of issuing gold coins therefore depends heavily on both \(P^*\) and \(P^+\). In particular, even if both these parameters are close to 1, the Government will find it profitable to mint gold coins so long as \(P^*\) is significantly larger than \(P^+\). Figure III plots the Government’s gain as a function of the liquidity parameters, on the assumption that the Government chooses the gold content which maximises their profit (or minimises their gain) per quantum of gold used. If \(P^*\) is 1.05, \(P^+\) has to exceed 1.108 if the authorities are to recover the cost of the gold from seignorage gains. If \(P^*\) is 1.01 (1.001) the comparable figure for \(P^+\) is only 0.019 (0.006).

Figure IV shows the gains to households as a function of the gold value of the coins. In the case where \(P^* = P^+ = 1\), the coins have no liquidity value, and issuing coins has a cost because gold is tied up in a non-productive use. If liquidity shocks are significant, there are substantial benefits from the issue of gold coins, so that when \(P^* = P^+ = 1.05\), there is a net gain to households across a range of initial levels of bullion content. If large shocks are also severe shocks \((P^* = 1.05, P^+ = 1.10)\), the gains to households may amount to 10% or more of the value of the gold used.

\textbf{V. Conclusions}

In this paper we have investigated the effect of introducing commodity money into an economy with a fiat currency. We have shown that commodity money can play a distinct role in the provision of liquidity services, being less costly to hold but more costly to use than ordinary cash. This makes it particularly suitable for meeting liquidity shocks which
are large and severe. It is only a partial substitute for paper money; the demand for paper money falls, but there is an increase in the demand for money in total.

One important motivation for looking at the question is the existence of large stocks of gold in the hands of the monetary authorities which serve no very obvious purpose, but which they are reluctant to sell in the open market. Our main result is that the cost to Government of minting the gold rather than selling it depends heavily on whether large liquidity shocks also tend to be severe shocks. Where the severity of shocks and their size are independent, the cost of the gold exceeds the gains to Government from the increase in the stock of non-interest bearing money. However, if large liquidity shocks tend to be severe shocks, gains from seignorage may outweigh the value of the gold used.

With the assumption of independence between the severity of shocks and their size, we found that the net cost to Government of issuing gold coins could be decomposed into two sources. The Government issues coins at par rather than at market value, and the Government forgoes lease income on the gold while it is used in the coins.

We also looked at the wider gains to households from the issue of gold coins, noting that these include not only any benefits to Government (which are assumed to be passed on to households in this model through a non-distorting tax rebate), but also the windfall gains to those households who acquire valuable coins at face value either from Government at the mint or from creditors who choose to discharge their debts using gold coin. The net gain to households to be composed of two offsetting elements: the benefits from being able to meet liquidity shocks more fully, offset by the loss of lease income from the gold in the coins.

We were able to parameterise the model to provide estimates of the magnitudes of the effects involved. We showed that, with the independence assumption, the net cost to Government of issuing gold coins might be of the order of 10% of the value of gold used. If liquidity shocks are sufficiently costly, there is a net gain to households. By relaxing the independence assumption, we found plausible parameter values which turn the cost to Government into a net benefit. The Government can make money from minting gold
coins even if the cost of meeting liquidity shocks is low provided only that large shocks are substantially more costly to meet than small shocks.

These results suggest that the use of gold for minting coins merits serious consideration. We have shown that the cost to Government of issuing gold coins is largely due to the loss of leasing income on gold that is used in coins. We therefore probably overstate the true cost to Government since little of the gold held by Governments is in fact lent out.
Appendix A

Proof of Proposition 1

First we prove part (iii). Suppose the contrary, and that $G^* = \text{Max}\{G \mid P(G) = 1\}$. At this gold price, gold coins dominate cash because they cost the same and the coins may appreciate. So everyone would want to hold the coins rather than cash. Demand for money overall can be no lower with gold coins than it would be without, so the demand for gold coins is at least equal to $q^*$. But the supply, $k^*$ is less than this by assumption. So there is a contradiction.

Since $P > 1$, all coins must be in the hands of households ($q^a = k^a$) or are melted down, since any coins left with the authorities can be purchased at face value and sold at a profit. Hence part (ii). With $P > 1$, it is cheaper to use cash rather than coins when facing a liquidity shock, so cash is used first. Hence part (iv).

Define:

$$Y(u) = \begin{array}{l}
\int_{c=1}^{\infty} \int_{q=q^a}^{\infty} \{\text{Min}(c,P)-1\} \lambda(q,c)dqdc + \int_{c=P+q^a+k^a}^{\infty} \{c-P\} \lambda(q,c)dqdc \\
- \int_{c=1}^{\infty} \int_{q=q^*}^{\infty} \{c-1\} \lambda(q,c)dqdc.
\end{array}
$$

(A-1)

From equations (3), (7a) and (7b), if the first order conditions are to be satisfied then $Y(q^c) = 0$. It can readily be verified that $dY/du < 0$, and $Y(q^* - k^a) > 0 > Y(q^*)$. So for any given $P > 1$, there is a unique $q^c \in (q^* - k^a, q^*)$ which satisfies the first order conditions. Hence parts (i) and (vi). This then uniquely defines $r^c \in (0, r)$, which is part (v). Equation (8) is just a rewriting of the equation $Y(q^c) = 0$, hence part (viii).

Finally, we need to show that there is some finite gold price $G^M$ above which it is best to melt the coins and extract the gold. We have stipulated that for an equilibrium $P(G)$ must tend to $G$ for large $G$. If there is no finite price $G^M$ at which equality occurs then $P(G)$ must get arbitrarily close to $G$. Consider a strategy of buying one coin and selling gold forward. For large $G$ this is a strategy which has an arbitrarily small net investment or risk to capital. The flow of liquidity benefits from the gold coin cannot exceed:
\[
\int_{q=1}^{\infty} \int_{c=G}^{\infty} \{c - G\lambda(q, c)\} dq dc,
\]
which tends to 0 as \(G\) tends to infinity.

However the cost of borrowing the gold is \(lG\). So for large enough \(G\) it is better to hold gold rather than coins – but then the demand for coins goes to zero. This gives part (vii).

**Proof of Proposition 2**

Consider states of the world where \(P > 1\). If there is an internal solution, then it must satisfy \(Y(q^c) = 0\). \(dY/dq\) is still negative. From (A-1) we can write \(Y(0)\) as:

\[
Y(0) = \int_{q=1}^{q^*} \int_{c=0}^{\infty} \{c - 1\lambda(q, c)\} dq dc - \int_{q=p}^{q^*} \int_{c=0}^{\infty} \{c - P\lambda(q, c)\} dq dc.
\]

(A-3)

Since \(k^g\) is now assumed to be greater than \(q^*\), if \(P\) is sufficiently close to 1, \(Y(0) < 0\). There is then no feasible internal solution. So for \(P\) close to but greater than 1, \(q^c = 0\). If \(P > 1\), all coins must be with households so \(q^g = k^g\) and \(r^g\) is then given by:

\[
r^g = \int_{c=p}^{\infty} \int_{q=k^g}^{q^*} \{c - P\lambda(q, c)\} dq dc.
\]

(A-4)

\(r^g \in (0, r)\). For higher values of \(P\), \(Y(q^c) = 0\) does have an internal solution, and the arguments of Proposition 1 apply. This covers the medium and high gold price regions.

If \(P = 1\) and \(r^g = r\) gold coins and cash are interchangeable. The total volume of cash held by households is less than the issue of gold coins, so some coins remain with the authorities and this keeps their price down to par.

It remains to make sure that the boundary between the low and medium gold price region is feasible. \(P(G)\) is continuous, and so is its first derivative at the boundary. But the second derivative is not continuous. Below the boundary the opportunity cost of holding gold coins is \(r\). The total amount of cash held is \(q^*\). Immediately above the boundary, (A-4) shows that the opportunity cost falls to:
This step change in the opportunity cost of gold coins means that the optimal holding jumps from \( q^* \) to \( k^g \).

**Proof of Proposition 3**

Take the limit of (10) as \( k^g \) tends to 0, and substitute from (9) to get:

\[
\Pi^G = 1 - G_0 - E\left[ \int_0^T r \frac{\int_{c=P}^\infty \left(c - P\right) \lambda(q^*, c) d\lambda}{\int_{c=1}^\infty \left(c - 1\right) \lambda(q^*, c) d\lambda} e^{-rt} dt \right],
\]

(A-6)

where \( T \) is the random time at which the price of gold first reaches \( G^M \), and the coins are melted down. From equations (7a) and (7b):

\[
\text{As } k^g \to 0, \quad r^g \to \int_{c=P}^\infty \int_{q=q^*}^\infty \left\{c - P\right\} \lambda(q, c) dq dc
\]

and

\[
\frac{r^g}{r} \to \int_{c=1}^\infty \int_{q=q^*}^\infty \left\{c - 1\right\} \lambda(q, c) dq dc.
\]

(A-7)

If \( c \) and \( e \) are independent, the ratios of conditional expectations in (A-6) and (A-7) are equal, so the gains to Government are:

\[
\Pi^G = E\left[ 1 - G_0 - \int_{t=0}^T r_t^g e^{-rt} dt \right]
\]

\[
= E\left[ 1 - G_0 - \int_{t=0}^T \left( rP_t - E[ \frac{dP_t}{dt} e^{-rt} dt] \right) e^{-rt} dt \right]
\]

(A-8)

\[
= \{P_0 - 1\} - \{G_0 - G^M E[e^{-rt}] \}.
\]
The second line of (A-8) uses the definition of \( r^g \) in (5), and the third line is obtained by integrating by parts, and using the definition of \( T \) as the random time at which the coin is melted down and is worth \( G^M \).

The final step is to note that a strategy of buying a unit of gold at time 0, leasing it and then selling it at time \( T \) has a present value of zero, so:

\[
-G_0 + \mathbb{E}\left[ \int_0^T l G e^{-\tau} \, dt \right] + \mathbb{E}[G^M e^{-rT}] = 0. \tag{A-9}
\]

The result immediately follows.

**Proof of Proposition 5**

In the limit case where the gold coin issue is small, \( P \) is a function of \( G \) alone, and \( G \) follows a geometric diffusion. We can use the Martingale Representation Theorem to rewrite the expectation in (5) as a derivative, and get:

\[
\Delta P = -r^g, \tag{A-10}
\]

where \( \Delta P \equiv \frac{1}{2} \sigma^2 G^2 \frac{d^2P}{dG^2} + (r - l)G \frac{dP}{dG} - rP. \)

It applies so long as the coin is in circulation. For \( G \geq G^M \) the value of the coin is its gold content, so \( P = G \). To avoid infinite instantaneous expected returns, it is necessary that \( P \) be twice differentiable for all \( G \in (0, \infty) \). The boundary conditions on \( P \) are:

\[
P \to 1 \text{ as } G \to 0, \text{ and } P \to G \text{ and } dP/dG \to 1 \text{ as } G \to G^M. \tag{A-11}
\]

Assumptions \([A6]\) and \([A7]\) imply that \( \lambda \) can be written as:

\[
\lambda(q, c) = A \exp\left[ -q/a \right] \delta(c - Bq - D), \tag{A-12}
\]

where \( \delta \) is the Dirac delta function, and \( A, B, D \) and \( a \) are positive constants. Substitute (A-12) into (9) to get:
\[
\lim_{k^* \to 0} \left[ \frac{q^*-q^c}{k^*} \right] = \frac{\max[P^*-P,0]}{P^*-1},
\]
\[\text{(A-13)}\]
where \(P^* \equiv E[c|q = q^*] = D + Bq^*\).

Substitute (A-12) into (A-7) to get:
\[
\frac{r^*}{r} = \begin{cases} 
\frac{P^*-P}{P^*-1} & \text{if } P \leq P^* \\
(P^*-P^*) \exp \left[ \frac{(P^*-P)/(P^*-1)}{P^*-1} \right] & \text{otherwise}
\end{cases}
\]
\[\text{(A-14)}\]
where \(P^* \equiv E[c|q > q^*] = D + B(q^* + a)\).

The differential equation for \(P\) is (A-10) with \(r^*\) substituted from (A-14).

To get the equation for \(\Pi^G\), apply the Martingale Representation Theorem to (10), taking the limit as \(k^*\) tends to zero, to get:
\[
\Delta \Pi^G = lG - r + \lim \left[ \frac{r q^*-q^c}{k^*} \right].
\]
\[\text{(A-15)}\]
Substitute the limit expression from (A-13) and the result follows. The conditions for \(G = 0\) and \(G > G^M\) follow directly from Proposition 3.

Reasoning similarly, from equation (13) \(\Pi^H\) satisfies the differential equation:
\[
\Delta \Pi^H = \Delta \Pi^G + \Delta P + r - (P-1) \int_{c=p}^{\infty} \int_{q=q^*}^{\infty} \lambda(q,c)dqdc.
\]
\[\text{(A-16)}\]
Substituting from (A-12) we get:
\[
\int_{c=p}^{\infty} \int_{q=q^*}^{\infty} \lambda(q,c)dqdc = \begin{cases} 
\frac{r}{P^*-1} & \text{if } P \leq P^* \\
\frac{r \exp[(P^*-P)/(P^*-P^*)]}{P^*-1} & \text{if } P > P^*.
\end{cases}
\]
\[\text{(A-17)}\]
The differential equation for \(\Pi^H\) then results by substituting (A-17) into (A-16).
Appendix B

*Equivalence of Flow of Liquidity Services model*

An alternative approach to modelling demand for liquidity is to postulate that money generates a flow of liquidity services which we can denote by $L$. The objective of the household is to allocate its financial assets between bonds and different forms of cash so as to maximise the sum of financial returns and liquidity services. We show here that by making suitable assumptions about the function $L$ this approach can be made identical to the approach followed in the body of the paper.

The flow of liquidity services depends on the household’s cash holdings $q^c$, its total holdings of legal tender $q^c + q^g$, and the price of gold coins, $P$. So we postulate that:

$$ [B1] \quad L = L(q^c, q^c + q^g, P). $$

If the household has neither cash nor coins, it gets no flow of liquidity services:

$$ [B2] \quad L(0, 0, P) = 0. $$

When the price of gold coins is high, they have little liquidity value:

$$ [B3] \quad L(q^c, q^c + q^g, P) \to L(q^c, q^c, P) \text{ as } P \to \infty. $$

If the household has no gold coins, the price of gold coins does not affect $L$:

$$ [B4] \quad L_\gamma(q^c, q^c, \cdot) = 0, $$

where the subscript denotes the partial derivative.

The liquidity gain from increasing the amount of cash and simultaneously reducing the amount of coins held is $L_\gamma$. This is in general positive. If we assume that the gain is independent of the amount of coin held (because cash is always used before coin), we have:
Finally, we make a number of assumptions about asymptotic behaviour and the sign of derivatives which are plausible:

\[ B6 \quad L_{13} \to 0 \text{ as } P \text{ or } q^c \to \infty, \text{ and } L_{11,13} \geq 0. \]

With these assumptions, it can be demonstrated that the liquidity function \( L \) must take the form:

\[
L = \int_{q=0}^{\infty} \int_{c=0}^{\infty} \min(q, q^c)(c-1)\lambda(q, c) dq dc \\
+ \int_{q=0}^{\infty} \int_{c=0}^{\infty} \min(q - q^c, q^g)(c-P)\lambda(q, c) dq dc,
\]

where \( \lambda(q, c) \geq 0. \) But this can be written as:

\[
L = \int_{q=0}^{\infty} \int_{c=0}^{\infty} [qc - f(q, c; q^c, q^g)]\lambda(q, c) dq dc,
\]

where \( f \) is defined in equation (4) and is the cost of meeting the liquidity shock \((q, c)\) when hold \( q^c \) in cash and \( q^g \) in coins. Since the cost of meeting the liquidity shock with no money is \( qc \), (B-2) does indeed measure the expected flow of liquidity benefits from holding money in the model.
Figure I: The Market Value of a Gold Coin as a function of its Gold content

The figure shows the market value of a gold coin (expressed as a proportion of its face value) varies with its gold content (expressed as the ratio of the bullion value of the coin to its face value). The three curved lines show the value function for different values of the liquidity parameters $P^*$ and $P^+$. $P^*$ is the average severity of the largest liquidity shock which can be met in full by the average household. $P^+$ is the average severity of shocks whose magnitude cannot be met in full. The risk free rate is 5%, the lease rate is 1% and the volatility of the gold price is 15%, all figures being annual rates.
Figure II: Gain to Government from the issue of Gold Coins

The figure shows the gain to Government from a small issue of gold coins, expressed as a percentage of the value of the gold used. With all the points lying below the zero axis, this shows that for the parameter values selected the issue has a net cost to Government. The gain is shown as a function of the gold content of the coins (the bullion value to face value ratio). The three lines relate to different values of the liquidity parameters $P^*$ and $P^+$. $P^*$ is the average severity of the largest liquidity shock which can be met in full by the average household. $P^+$ is the average severity of shocks whose magnitude cannot be met in full. The risk free rate is 5%, the lease rate is 1% and the volatility of the gold price is 15%, all figures being annual rates.
The figure shows the gain or cost to the Government from minting coins, for different liquidity parameters. The Government is assumed to decide on how much gold it wishes to use for coins, and then determines the ratio of gold value to face value to maximise its gains or minimise its losses. The three lines relate to different values of the liquidity parameters $P^*$. $P^*$ is the average severity of the largest liquidity shock which can be met in full by the average household. The gain is plotted against $P^+ - P^*$ where $P^+$ is the average severity of shocks whose magnitude cannot be met in full. The risk free rate is 5%, the lease rate is 1% and the volatility of the gold price is 15%, all figures being annual rates.

Figure III: Gains to Government from Coin issue with Optimal Gold Content
Figure IV: Gains to Households from Issuing Gold Coins

The figure shows the gain to Households from a small issue of gold coins, expressed as a percentage of the value of the gold used. The gains include the gain or loss to Governments from the issue which is returned to households in the form of a tax or tax rebate. The gain is shown as a function of the gold content of the coins (the bullion value to face value ratio). The three lines relate to different values of the liquidity parameters $P^*$ and $P^+$. $P^*$ is the average severity of the largest liquidity shock which can be met in full by the average household. $P^+$ is the average severity of shocks whose magnitude cannot be met in full. The risk free rate is 5%, the lease rate is 1% and the volatility of the gold price is 15%, all figures being annual rates.
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